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Rayleigh-Ritz Approximation for the Stability of Localised Waves

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Abstract. We consider fundamental localised waves of the ϕ^4 equation. Using the variational principle, i.e., Rayleigh-Ritz method, we solve the corresponding eigenvalue problem of the waves and compute spectrum of the linear spectral operator. By comparing with numerical computations, we show that our approximation has better agreement than existing results in a wide range of coupling constant.

Keywords: solitons, variational approximation, ϕ^4-equation, spectrum

INTRODUCTION

We consider the ϕ^4 equation

$$\dot{u}_n = c(u_{n+1} - 2u_n + u_{n-1}) + u_n^3 - u_n, \tag{1}$$

where u_n is a real-valued wave function at site $n \in \mathbb{Z}$ and $c > 0$ is the strength of the coupling between adjacent sites which is also called as the dispersion coefficient. In the continuum limit $c \to \infty$, the hyperbolic version of the equation belongs to the class of nonlinear Klein-Gordon equations that appears as a model in field theory [1]. Equation (1) can also be seen as a small amplitude approximation of the sine-Gordon equation [2]. In this paper, as we are interested in time-independent solitary waves and their stability, we consider the parabolic version (overdamped) of the equation.

If v_n is the stationary solution of (1), then it will satisfy the time-independent equation,

$$v_n = c(v_{n+1} - 2v_n + v_{n-1}) + v_n^3. \tag{2}$$

Equation (2) admits two fundamental discrete solitary waves, which interchangeably will also be called discrete solitons, that can be continued all the way from the uncoupled limit $c \to 0$ to the continuous limit $c \to \infty$. The two solutions are on-site (i.e., site-centred) and inter-site (i.e., bond-centred) discrete solitons, which are also usually referred to as Sievers-Takeno (ST) and Page (P) mode, respectively. The two modes become degenerate in the continuous limit $c \to \infty$.

There is no explicit analytical expression of the fundamental solitons. Four different approximations for the discrete solitons have been considered before, namely a variational approach, an approximation to homoclinic orbits, a Green-function approach, and a quasi-continuum approximation [3].

After a discrete soliton is obtained, the following natural question would be to determine its stability by computing the spectrum of the corresponding linear spectral operator. Analytical methods to solve the eigenvalue problem have been an Evans function method [4, 5], an anti-continuum perturbation expansion [6, 7], and stability of the variational parameters [8, 9].

In this paper, we aim to apply the variational principle, also known as Rayleigh-Ritz method [10], to solve the corresponding eigenvalue problem of the fundamental discrete solitons. As for the soliton solutions that will determine the potential of the eigenvalue problem, we use the variational method as approximations. Our result can approximate

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eigenvalues bifurcating from the continuous spectrum when the coupling constant is varied. By comparing the results with numerical computation, we obtained good agreement.

LOCALISED WAVES

We will approximate the discrete soliton profile analytically using variational approximation. We begin by noticing that the Lagrangian of the system (1) is given by

$$L = \sum_{n=-\infty}^{\infty} \frac{1}{2}(1+2c)v_n^2 - \frac{1}{4}v_n^4 - \frac{c}{2}(v_{n+1}+v_{n-1})v_n. \tag{3}$$

We can observe that (2) is the minimizer of the Lagrangian (3).

Now, introduce the 'soliton-like' ansatz as an approximation

$$v_n^{ST} = A_1 e^{-a_1|n|}, \tag{4}$$

$$v_n^{P} = A_2 e^{-a_2|n-\frac{1}{2}|}, \tag{5}$$

where v_n^{ST} denotes soliton in ST-modes (on-site soliton) and v_n^P in P-modes (inter-site soliton), A_1, a_1, A_2, a_2 are variational parameters to be determined [3]. Substituting these ansatzs to the Lagrangian will yields effective Lagrangians. For the first ansatz (4), we get the following

$$L_{eff}^{ST} = \frac{A_1^2\left((A_1^2-4c-2)e^{4a_1} + 8ce^{3a_1} - (8c+4)e^{2a_1} + 8ce^{a_1} + A_1^2 - 4c - 2\right)}{4-4e^{4a_1}}. \tag{6}$$

While for the second ansatz (5), we get the following effective Lagrangian

$$L_{eff}^{P} = \frac{A_2^2\left((A_2^2+4c)e^{2a_2} - (2c+2)e^{3a_2} - 2ce^{-a_2} - (4c+2)e^{a_2} + 4c\right)}{2-2e^{4a_2}}. \tag{7}$$

Each of the effective Lagrangian is a function of variational parameters. Using variational principle, we can find the variational parameters A_1, a_1, A_2, a_2 by solving the following Euler-Lagrange equations

$$\frac{\partial L_{eff}^{ST}}{\partial A_1} = \frac{\partial L_{eff}^{ST}}{\partial a_1} = 0, \quad \frac{\partial L_{eff}^{P}}{\partial A_2} = \frac{\partial L_{eff}^{P}}{\partial a_2} = 0. \tag{8}$$

The equations to be solved are then

$$(A_1^2 - 2c - 1)e^{4a_1} + 4ce^{3a_1} - (4c+2)e^{2a_1} + A_1^2 + 4ce_1^a - 2c - 1 = 0, \tag{9}$$

$$ce^{6a_1} - (2c+1)e^{5a_1} + 3ce^{4a_1} + (A_1^2 - 4c - 2)e^{3a_1} + 3ce^{2a_1} - (2c+1)e_1^a + c = 0, \tag{10}$$

$$-(c+1)e^{3a_2} + (A_2^+ 2c)e^{2a_2} - (2c+1)e^{a_2} - ce^{-a_2} - c - 2c = 0, \tag{11}$$

$$\begin{aligned} -(c+1)e^{7a_2} + (A_2^2+4c)e^{6a_2} - (6c+3)e^{5a_2} + 8ce^{4a_2} - (8c+3)e^{3a_2} & \\ +(A_2^2+4c)e^{2a_2} - (2c+1)e_2^a + ce^{-a_2} &= 0. \end{aligned} \tag{12}$$

Substituting back the values of the parameters obtained from (8) to (4) and (5), we can then obtain an approximate solution of (2). Figure 1 shows the profiles of on-site and inter-site solitons for several values of c. We can see that the obtained approximations are quite close to the numerical results.

EIGENVALUES

In this section we will compute the eigenvalues of the two solitary waves. Linear stability of a standing wave of (1) is determined by solving a corresponding eigenvalue problem that can be derived as follows.

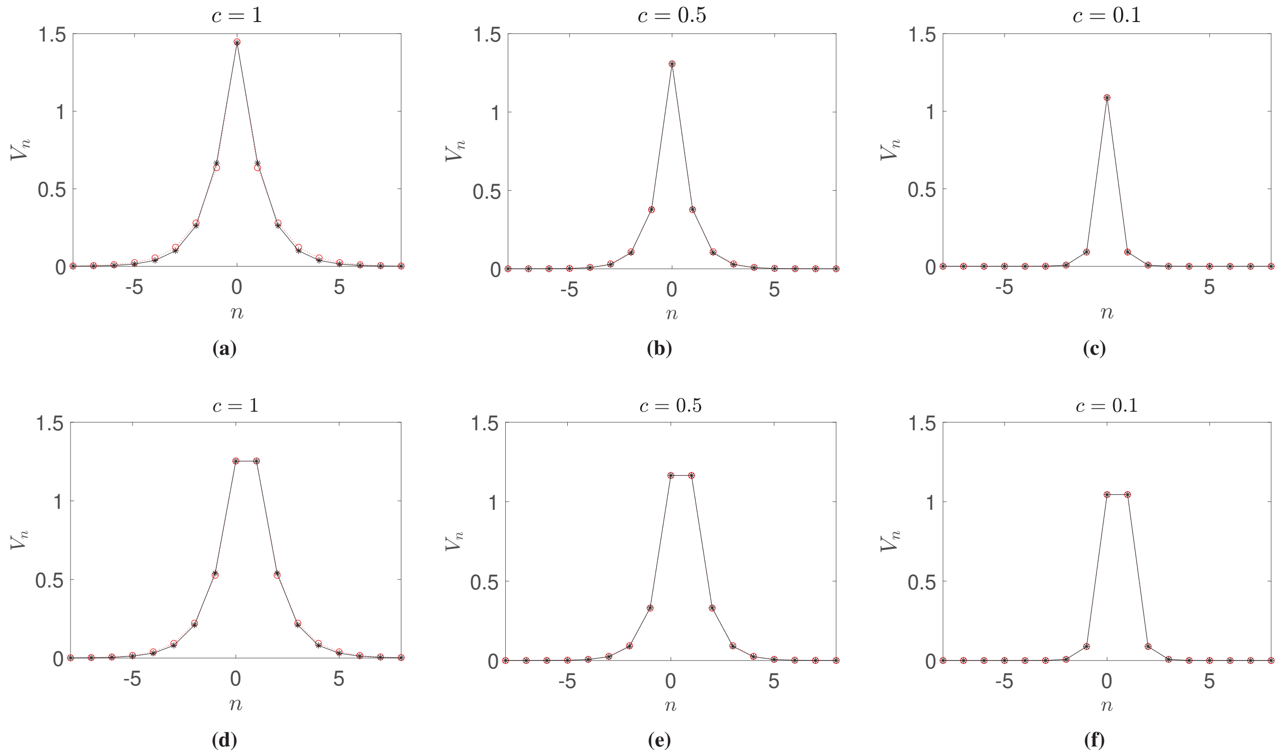


FIGURE 1. Profiles on-site (a-c) and inter-site (d-f) discrete solitons of (1) for different values of c as indicated. Stars and circles are numerical computations and approximations, respectively.

Introduce the linearisation ansats $u_n = v_n + \delta\epsilon_n(t)$, where v_n is the stationary solution satisfying (2) and $\delta \ll 1$. Substitute this into (1) to yield the linear equation

$$\dot{\epsilon_n} = c(\epsilon_{n+1} - 2\epsilon_n + \epsilon_{n-1}) + 3v_n^2\epsilon_n - \epsilon_n, \tag{13}$$

where neglect higher order terms.

Now, write $\epsilon_n = \eta_n e^{\lambda t}$, plug into the last equation and simplify it to obtain

$$\lambda\eta_n = c(\eta_{n+1} - 2\eta_n + \eta_{n-1}) + 3v_n^2\eta_n - \eta_n. \tag{14}$$

Equation (14) is an eigenvalue problem for (1). The linear stability of discrete solitary waves of (1) is determined by solving this eigenvalue problem. The parameter $\lambda \in \mathbb{R}$ is the spectral parameter. The numerically computed structure of the spectrum of the ST and P modes are presented in Figure 2. The figure shows that the spectrum of the modes consists of two parts, namely continuous and discrete spectrum (eigenvalue).

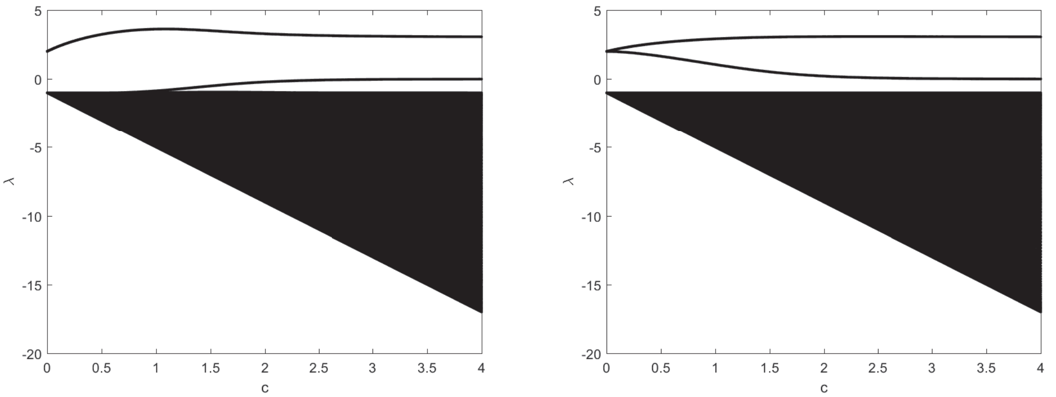
The continuous spectrum can be obtained by substituting $\eta_n = e^{ikn}, k \in \mathbb{R}$ in (14) for $n \to \infty$

$$\lambda = 2c(\cos k - 1) - 1.$$

Therefore, the continuous spectrum is given by $-4c - 1 < \lambda < -1$.

Discrete spectrum or eigenvalue depends on the shape of the potential $\sim v_n^2$ that depends on the coupling constant c. When $c = 0$, there is only one eigenvalue, $\lambda = 2$ for both modes. If we zoom out Figure 2, for the on-site modes, when c increases, there will be two eigenvalues bifurcating from the continuous spectrum at $c \approx 0.6$ and $c \approx 0.9$. While for the inter-site modes, there will be one eigenvalue bifurcating from the continuous spectrum at $c \approx 2.2$.

We are going to approximate the discrete spectrum using Rayleigh-Ritz method, that uses the following fact. Given an eigenfunction η_n, we can find the explicit formula of λ, that is by taking the inner product both sides of the



(a) **(b)**

FIGURE 2. Spectrum of the (a) ST and (b) P modes of (1) for varying c.

equation (14) with η_n, i.e.,

$$\lambda = \frac{\left\langle c(\eta_{n+1} - 2\eta_n + \eta_{n-1}) + 3v_n^2\eta_n - \eta_n, \eta_n \right\rangle}{\langle \eta_n, \eta_n \rangle}, \tag{15}$$

where the inner product is defined as $\langle \eta_m, \eta_n \rangle = \sum_{n,m=-\infty}^{\infty} \eta_m \eta_n$. As we do not have an exact solution for the eigenfunction v_n, Rayleigh-Ritz method suggests us to use an approximation.

On-site solitary waves

As explained above, on-site solitary waves have three eigenvalues. One of them already exists from $c = 0$ and the other two bifurcate from the continuous spectrum.

Largest eigenvalue

To approximate the eigenfunction corresponding to the largest spectrum, we use the ansatz $\eta_n^{(1)} = e^{-\tilde{a}|n|}$, where $\tilde{a}$ is the variational parameter to be determined. Substituting this ansatz to equation (15) and using the on-site soliton (4), we obtain the eigenvalue

$$\lambda_1^{ST} = \frac{\left(-\frac{2c(1+e^{2\tilde{a}})}{e^{2\tilde{a}}-1} - \frac{4ce^{-\tilde{a}}}{e^{-2\tilde{a}}-1} - \frac{1+e^{2\tilde{a}}}{e^{2\tilde{a}}-1)} + \frac{3A_1^2(1+e^{2a_1+2\tilde{a}})}{e^{2\tilde{a}+2a_1}-1}\right)(e^{2\tilde{a}} - 1)}{1 + e^{2\tilde{a}}}. \tag{16}$$

According to Rayleigh-Ritz method, we can find $\tilde{a}$ that yields the best approximation of the eigenvalue from solving the Euler-Lagrange equation $\frac{d\lambda_1^{ST}}{d\tilde{a}} = 0$. Our approximate λ_1^{ST} is shown by the dashed red line in Figure 3(a).

Eigenvalues bifurcating from the continuous spectrum

Let $\eta_n^{(2)} = \text{sgn}(n)e^{-\tilde{a}|n|}$ and $\eta_n^{(3)} = \begin{cases} e^{-\tilde{a}|n|}, & n \neq 0, \\ B, & n = 0 \end{cases}$ be ansatz of the eigenfunctions corresponding to our second and third eigenvalues that bifurcate from the continuous spectrum, respectively. Using the same procedure as before, we obtain the eigenvalues corresponding to $\eta_n^{(2)}$ and $\eta_n^{(3)}$, respectively as follows

$$\lambda_2^{ST} = \frac{1}{2}\left(-\frac{4c}{e^{2\tilde{a}}-1} - \frac{ce^{-\tilde{a}}\left(3e^{-2\tilde{a}}-1\right)}{e^{-2\tilde{a}}-1} - \frac{ce^{-\tilde{a}}\left(e^{-2\tilde{a}}+1\right)}{e^{-2\tilde{a}}-1} - \frac{2}{e^{2\tilde{a}}-1} + \frac{6A_1^2}{e^{2a_1+2\tilde{a}}-1}\right)(e^{2\tilde{a}} - 1), \tag{17}$$

$$\lambda_3^{ST} = \left(-\frac{2c(A_1^2e^{2\tilde{a}}-A_1^2+2)}{e^{2\tilde{a}}-1} + \frac{4ce^{-\tilde{a}}(A_1e^{-2\tilde{a}}-e^{-2\tilde{a}}-A_1)}{e^{-2\tilde{a}}-1} - \frac{A_1^2e^{2\tilde{a}}-A_1^2+2}{e^{2\tilde{a}}-1} + \frac{3A_1^2(A_1^2e^{2a_1+2\tilde{a}}-A_1^2+2)}{e^{2a_1+2\tilde{a}}-1}\right)\frac{(e^{2\tilde{a}}-1)}{A_1^2e^{2\tilde{a}}-A_1^2+2}. \tag{18}$$

The dashed red lines in Figure 3(b) and Figure 3(c) show the graphs of λ_2^{ST} and λ_3^{ST} respectively for various values of c.

Inter-site solitary waves

The next type of our solitary waves has three eigenvalues. Two of them already exist from $c = 0$ and the other one bifurcates later on from the continuous spectrum.

Largest eigenvalues

Let $\eta_n^{(1)} = e^{-\tilde{a}|n-\frac{1}{2}|}$ and $\eta_n^{(2)} = \text{sgn}(n - \frac{1}{2})e^{-\tilde{a}|n-\frac{1}{2}|}$ be the corresponding eigenfunctions for the first and second largest discrete spectrum, respectively. The eigenvalues are then given by

$$\lambda_1^P = \frac{1}{2}\frac{\left(\frac{-4ce^{\tilde{a}}}{e^{2\tilde{a}}-1} + \frac{2c(e^{\tilde{a}}-e^{-\tilde{a}}+2)}{e^{2\tilde{a}}-1} - \frac{2e^{\tilde{a}}}{e^{2\tilde{a}}-1} + \frac{6A_2^2e^{a_2+\tilde{a}}}{e^{2a_2+2\tilde{a}}-1}\right)(e^{2\tilde{a}} - 1)}{e^{\tilde{a}}}, \tag{19}$$

$$\lambda_2^P = \frac{e^{-\tilde{a}}(2ce^{2a_2+2\tilde{a}} + 3A_2^2e^{a_2+3\tilde{a}} - 2c - 3ce^{2a_2+3\tilde{a}} - 3A_2^2e^{a_2+\tilde{a}} - e^{2a_2+3\tilde{a}} + 3ce^{\tilde{a}} + ce^{2a_2+\tilde{a}} + e^{\tilde{a}} - ce^{-\tilde{a}}}{(e^{2a_2+2\tilde{a}} - 1)}. \tag{20}$$

The dashed red lines in Figure 4(a) and Figure 4(b) show the graphs of λ_1^P and λ_2^P, respectively.

Eigenvalue bifurcating from the continuous spectrum

Let $\eta_n^{(3)} = \begin{cases} e^{-\tilde{a}|n-\frac{1}{2}|}, & n \neq 0, 1, \\ B_1, & n = 0, 1, \\ B_2, & n = -1, 2 \end{cases}$ be our ansatz for the eigenfunction of the eigenvalue bifurcating from the continuous spectrum. Using Rayleigh-Ritz method, we obtain the eigenvalue as

$$\begin{aligned}\lambda_3^P = \quad & \frac{1}{(e^{2\tilde{a}+2a_2} - 1)(B_1^2e^{2\tilde{a}} + B_2^2e^{2\tilde{a}} - B_1^2 - B_2^2 + e^{-3\tilde{a}})}\{2B_2e^{\frac{3}{2}\tilde{a}+2a_2}c + 3A_2^2B_2^2e^{-a_2+4\tilde{a}} + 3A_2^2B_1^2e^{a_2+4\tilde{a}} \\ & +3A_2^2B_2^2e^{-3a_2} - 2B_2e^{-\frac{1}{2}\tilde{a}}c - 2B_2e^{-\frac{1}{2}\tilde{a}+2a_2}c - 2B_2^2e^{4\tilde{a}+2a_2}c - 3A_2^2B_2^2e^{-3a_2+2\tilde{a}} - 3A_2^2B_2^2e^{-a_2+2\tilde{a}} \\ & -3A_2^2B_1^2e^{-a_2+2\tilde{a}} - 3A_2^2B_1^2e^{a_2+2\tilde{a}} - e^{-\tilde{a}+2a_2} + 2e^{-3\tilde{a}}c + 3A_2^2e^{-3a_2-\tilde{a}} - 3A_2^2e^{-3a_2-3\tilde{a}} - 2e^{-\tilde{a}+2a_2}c - B_1^2c \\ & -2B_2^2c + B_1^2e^{2\tilde{a}+2a_2} + B_2^2e^{2\tilde{a}+2a_2} - 2ce^{-4\tilde{a}} + B_1^2e^{2\tilde{a}} + B_2^2e^{2\tilde{a}} - B_2^2 - B_1^2 + e^{-3\tilde{a}} + 2ce^{-2\tilde{a}+2a_2} \\ & +2B_1B_2e^{4\tilde{a}+2a_2}c - B_1^2e^{4\tilde{a}+2a_2} - B_2^2e^{4\tilde{a}+2a_2} - 2B_1B_2e^{2\tilde{a}}c - 2B_1B_2e^{2\tilde{a}+2a_2}c + 2B_1B_2c + 3A_2^2B_1^2e^{-a_2} \\ & +B_1^2e^{2\tilde{a}}c + B_1^2e^{2\tilde{a}+2a_2}c + 2B_2^2e^{2\tilde{a}}c + 2B_2^2e^{2\tilde{a}+2a_2}c + 2B_2e^{-\frac{5}{2}\tilde{a}}c - B_1^2e^{4\tilde{a}+2a_2}c\}.\end{aligned} \tag{21}$$

We plot our approximation λ_3^P when parameter c is varied in Figure 4(c), shown by the dashed red line.

Figures 3 and 4 show that the Rayleigh-Ritz method that we proposed herein yields good agreement with numerical computations, even until $c \sim O(1)$.

CONCLUSIONS AND FUTURE WORK

We have studied fundamental stationary solitary waves of the overdamped ϕ^4 equation and their spectrum. We used analytical approximation based on the Rayleigh-Ritz method to approximate them. Comparing with numerical computations, our approximations showed good agreement in a wide interval of coupling parameter. Applying the proposed method herein to solitary waves of other known discrete equations, such as the discrete nonlinear Schrödinger equation, is proposed as a future work.

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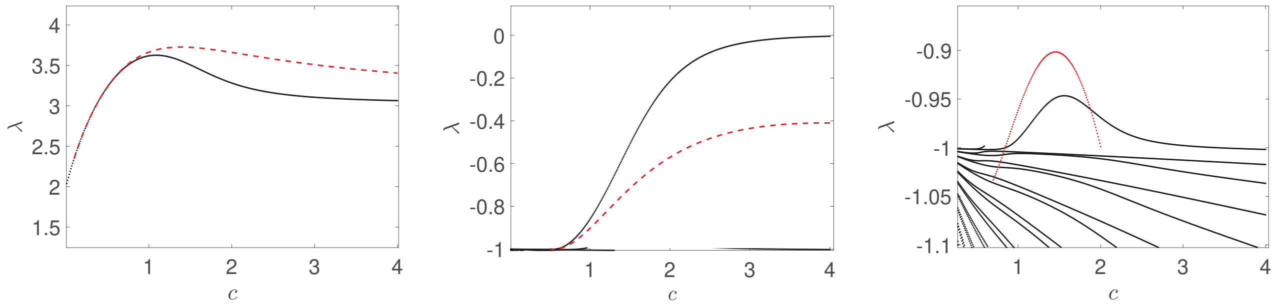
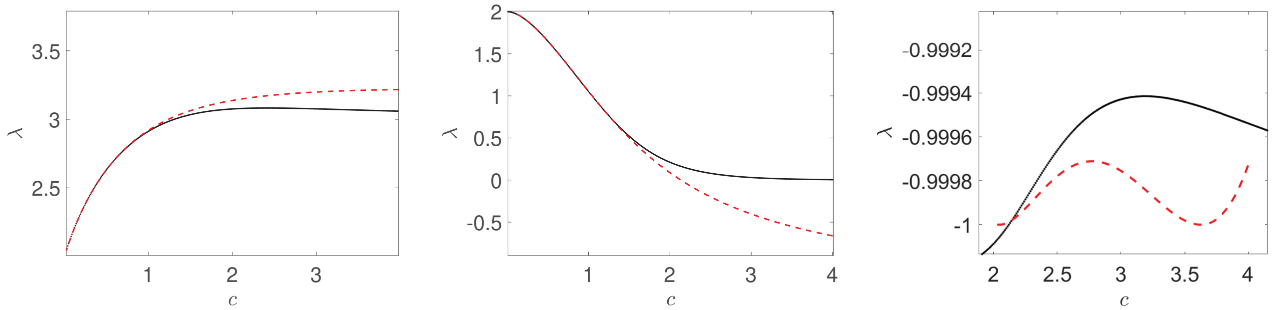
The authors contribute equally to the manuscript.



(a) (b) (c)

FIGURE 3. Spectrum of (14) as a function of coupling constant c for on-site solitary waves. Dashed red lines are the approximations using Rayleigh-Ritz method which in (a-c) is from (16), (17), and (18), respectively and black curves are from numerical computations .



(a) (b) (c)

FIGURE 4. The same as Figure 3, but for the inter-site solitary waves which in (a-c) is from (19), (20), and (21), respectively.

REFERENCES

[1] Benci, Vieri, and Donato Fortunato. "Variational methods in nonlinear field equations." Springer Monographs in Mathematics, Springer 2 (2014): 3.

[2] Cuevas-Maraver, Jesús, Panayotis G. Kevrekidis, and Floyd Williams. "The sine-Gordon model and its ap-plications." Nonlinear Systems and Complexity (Switzerland: Springer) (2014).

[3] Cuevas, Jesus and James, Guillaume and Kevrekidis, Panayotis G and Malomed, Boris A and Sanchez-Rey, Bernardo (2008) Approximation of solitons in the discrete NLS equation *Journal of Nonlinear Mathematical Physics* 15(sup3), 124 – 136.

[4] Kevrekidis, P. G., and C. K. R. T. Jones. "Bifurcation of internal solitary wave modes from the essential spectrum." Physical Review E 61.3 (2000): 3114.

[5] Kapitula, Todd, and Panayotis Kevrekidis. "Stability of waves in discrete systems." Nonlinearity 14.3 (2001): 533.

[6] Susanto, Hadi. "Localized modes and phonon scattering of a lattice κ kink." Physical Review E 73.2 (2006): 026608.

[7] Derks, G., Doelman, A., van Gils, S. A., and Susanto, H. (2007). Stability analysis of π-kinks in a 0-π Josephson junction. SIAM journal on applied dynamical systems, 6(1), 99-141.

[8] Chong, Christopher, and Dmitry Pelinovsky. "Variational approximations of bifurcations of asymmetric solitons in cubic-quintic nonlinear Schrödinger lattices." Discrete & Continuous Dynamical Systems-S 4.5 (2011): 1019-1031.

[9] Rusin, Rahmi, Rudy Kusdiantara, and Hadi Susanto. "Variational approximations using Gaussian ansatz, false instability, and its remedy in nonlinear Schrödinger lattices." Journal of Physics A: Mathematical and Theoretical 51.47 (2018): 475202.

[10] Capri, Anton Z. Nonrelativistic quantum mechanics. World Scientific Publishing Company, 2002.