Bifurcations in a Heterogeneous Agent Model

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For any errors or inadequacies that may remain in this work, of course, the responsibility is entirely my own.
Abstract

This paper investigates the dynamics in a simple asset pricing model with heterogeneous beliefs, which is based on a fundamental financial knowledge and detailed mathematical derivation of the fixed points and period doubling bifurcation. We concentrate on the relationship between increasing number of active traders and fluctuations of asset prices. The experiment result is that, when the total number of active traders increases, bifurcations and chaos are observed in the heterogeneous agent model.
Contents

1. Introduction 5

2. Description of logistic map and bifurcation 10
   2.1 Logistic family of mapping 10
   2.2 Behavior of the logistic map 11
   2.3 The period doubling bifurcation 16
      2.3.1 Logistic map of fixed point 16
      2.3.2 Logistic map of period-2 point 20

3. EMH and Capital Asset Pricing Model 24
   3.1 The Efficient Market Hypothesis 25
      3.1.1 Application of the EMH to Financial Markets 26
      3.1.2 Critique of the EMH 29
   3.2 The asset pricing model 30
      3.2.1 Derivation of the Capital Asset Pricing Model 30
      3.2.2 Critical analysis about CAPM 34

4. Asset pricing model with the heterogeneous beliefs and bifurcation 36
   4.1 The model 36
   4.2 Numerical analysis in the model 40

5. Conclusion 46

APPENDIX A 48

BIBLIOGRAPHY 49
List of Figures

Figure 1: Logistic behavior with “μ = 0.9, X₀ = 0.9” 12
Figure 2: Logistic behavior with “μ = 1.5, X₀ = 0.9” 13
Figure 3: Logistic behavior with “μ = 2.9, X₀ = 0.9” 13
Figure 4: Logistic behavior with “μ = 3.0, X₀ = 0.9” 13
Figure 5: Logistic behavior with “μ = 3.5, X₀ = 0.9” 14
Figure 6: Logistic behavior with “μ = 3.6, X₀ = 0.9” 14
Figure 7: Logistic behavior with “μ = 4.0, X₀ = 0.9” 15
Figure 8: Four logistic mapping:
\[ Q₁(μ = 1), Q₂(μ = 2), Q₃(μ = 3) \text{ and } Q₄(μ = 4) \] 18
Figure 9: A partial bifurcation diagram for \(0 < μ < 3\) 19
Figure 10: The logistic mapping \(Q^2₀\), for the period-2 points 21
Figure 11: A partial bifurcation diagram for the logistic map under \(0 < μ < 3.5\) 23
Figure 12: The bifurcation diagram 24
Figure 13: The behavior of dynamic price with “θN = 0.99” 41
Figure 14: The behavior of dynamic price with “θN = 1.28” 42
Figure 15: The behavior of dynamic price with “θN = 1.65” 42
Figure 16: The behavior of dynamic price with “θN = 2.00” 42
Figure 17: The bifurcation diagram 45
1. Introduction

In recent years, the studying of period doubling bifurcation has received a lot of attentions due to its potential applications to economics and financial areas. With the development of bifurcation phenomena in finance, a great impact has been imposed on financial market, especially in the asset pricing model. In finance, efficient market hypothesis (EMH) and capital asset pricing model (CAPM) are regarded as the cornerstone of financial theory in the past years. It has been recognized that some of the traditional linear models and financial hypothesis are deficient. For example, the capital asset pricing model (CAPM) assumes that all the investors have same expectations of asset returns. However, in realistic financial market, different investors have different beliefs of expectations and risk preferences. According to the viewpoint of Hsieh (1991), compared with linear models, nonlinear models can generate much richer types of behavior, such as bifurcation as well as chaos. Moreover, even simple nonlinear maps (like logistic map) can generate very complicated dynamics (May, 1976 cited in Strogatz, 1994). Therefore, an increasing number of financial economists began to lay emphasis on the
nonlinear financial model in order to find models, which can provide a better explanation.

According to David (1991, pp.1840), chaos can be defined as “a nonlinear deterministic process, which ‘looks’ random.” However, some mathematicians believed that chaotic behavior is not completely random. More specifically, Stutzer (1980) considers that this kind of randomness in chaotic behavior is caused by the factors within the system not external disturbances. This point of view seems partly in keeping with the principle of the efficient market hypothesis in finance. There are lots of papers about chaos in financial and economic world. For example, Lu et al. (2010) propose a conceptual framework with considering chaos fundamentally. Wang et al. (2010) believe that financial system is in a different state depends on the system parameters. Moreover, in Grandmont (1988), general mathematical facts of the bifurcations and guides research direction of economics towards chaotic systems are given. Due to the work of May (1976), bifurcation is one of the primary characteristics of the variables in the logistic map. Based on the empirical and numerical analysis, the logistic map gives a full interpretation of logistic behavior, from stable points through stable cycles (period doubling bifurcation)
to chaos. That is to say, bifurcation as one of the essential processes that is a prerequisite for emergence of chaos.

As mentioned before, there are two main principles: efficient market hypothesis (EMH) and capital asset pricing model (CAPM) have a great impact on financial markets. With the development of behavior finance, the applicability of EMH and CAPM are doubted in real financial market. The concept the efficient market hypothesis can be defined as “market in which prices always fully reflect available information” (Fama, 1970 pp. 383). It means when new information comes out, it will be incorporated into the asset price immediately and rationally without delay. However, Thaler (1987), De Long, Shleifer & Summers (1990) and Malkiel (2003), doubt that there are some conclusive evidences for the inapplicability of EMH, like, low volatility effect and under-reaction and over-reaction in new information.

Capital asset pricing model (CAPM) as the most-prominent model in asset pricing makes a great contribution to the commercial area. Based on a number of financial economists’ analysis in recent years, there is one of the most-attractive factors of CAPM is that “intuitively pleasing predictions about how to measure risk and about the relation between expected return and risk” (Fama and French, 2003, pp 1). In the Perold’s viewpoint (2004, pp.3), the CAPM is based on the idea of “not
all risks should affect asset pricing." Also, Andreas Krause (2001) gives the same explanation of CAPM, which is portfolio theory with a risk-free asset and unlimited short sales. However, due to the assumptions of EMH and CAPM that are very strict, it seems unrealistic to realize in real financial market. Based on the significant statistical findings and analyses of EMH and CAPM, an increasing number of economists find that the empirical result of the model is unsatisfactory. Thus, they began to combine the behavioral finance and CAPM in order to obtain a new model that is appropriate to the market. There are several well-known extension models in the financial market. For instance, eliminating the possibility of risk-free lending and borrowing (Black, 1972); adding international investing into model (Solnik, 1974; Stulz, 1981; Adler and Dumas, 1983); relying on the arbitrage pricing (Ross, 1976) as well as allowing the heterogeneous beliefs (Lintner, 1969; Merton, 1987). For the different extension of CAPM, there is different relationship and expectation between expected returns and risk.

Heterogeneity has been developed in various models in order to explain the financial market dynamic. An early example of heterogeneous agent models can be found in the economics and finance literature (see e.g. Zeeman, 1974; Day & Huang, 1990; Brock & de Fontnouvelle, 1996), which emphasize the heterogeneous beliefs
that may lead to dynamic and instability in a financial market, such as bifurcation and even chaos. Brock and Hommes (1998) investigate possible bifurcation routes to a dynamics asset pricing model with heterogeneous beliefs. These papers give a comprehensive understanding of heterogeneous agent model. Recent articles (such as Ricchiuti and Naimzada (2009), Kaizoji (2003) and Nikkhahan and Sohrabi (2011)) are more focused on the impact of specific parameters in the asset pricing model. Ricchiuti and Naimzada (2009) have analyzed the increasing distance between beliefs in the heterogeneous agent model that might give rise to bifurcation then chaos. Moreover, the paper of Kaizoji (2003) interprets how the total number of traders influences the fluctuations of asset prices in the heterogeneous agent model and find that when the amount of active traders increase chaos appeared.

In this paper, we aimed at using bifurcation theory and numerical methods to analyze fixed points and period doubling bifurcation existing in the asset pricing model with heterogeneous beliefs. There are three main contributions in this paper. Firstly, we use the graphic approach and the method of substitution to find the period doubling bifurcation in the asset pricing model. More specifically, in a heterogeneous agent model, the fluctuations of asset prices show a
range of behavior (like bifurcation and chaotic behavior) with different number of traders. Secondly, this paper provides essential financial background and mathematics method, which contribute to analyze the development of nonlinear asset pricing model. Thirdly, this paper gives a more detailed explanation of the existence of period doubling bifurcation in the heterogeneous agent model.

The remainder of this paper is organized as follows. Section 2 will give a graphical and algebraic description of logistic map and bifurcation. In section 3, a dialectical discussion of efficient market hypothesis is presented and introduces the theoretical background of capital asset pricing model. Section 4 presents the heterogeneous agent model briefly, and then, investigates the bifurcation of price dynamics. Finally, the conclusion is provided in section 5.

2. Description of logistic map and bifurcation

2.1 Logistic family of mapping

Logistic family of mappings is a standard example in non-linear dynamics system that is used to analyze chaotic behaviors over the last decade. Due to the work of May (1976) the logistic map as a
polynomial mapping has complicated dynamics and universality. The basic form of the logistic map is:

\[ X_{t+1} = \mu X_t (1 - X_t) \quad t = 0, 1, 2, \ldots \] (1)

where \( X_t \in [0, 1], \mu \in [0, 4]\)

In many applications (see e.g. May, 1976; Strogatz, 1994; Banks, Dragan & Jones, 2003) the logistic map is used to describe the growth of a seasonally breeding population. Also, \( X_t \) is the population of the \( t^{th} \) generation. That is to say, the size of the population cannot be negative; hence, the interval of \( X_t \) is between 0 and 1. For example, if the value of \( X_t \) is over than one that will lead to the iterations diverging towards \(-\infty\) and the population will die out (May, 1976). Moreover, according to May (1976) the changes of growth rate \( \mu \) will lead to the logistic equation exhibiting an astonishing range of behavior. Thus, in order to show how the fixed points change with a parameter on which the mapping depends on, in this section we firstly use the graphical approach to provide an overview of different dynamical behavior observed in the logistic map by plotting successive iterates under different \( \mu \). Then, algebraic method can be used to verify the result.

### 2.2 Behavior of the logistic map

We run the logistic map \( X_{t+1} = \mu X_t (1 - X_t) \) with initial value of
Figure 1: Logistic Behavior with “µ = 0.9, X₀ = 0.9”

X₀ = 0.9 and µ ∈ [0, 4], then the graphs plot the value of X₁, X₂, ..., X₁₀₀ under the different µ. Then, based on the different behavior that the logistic equation present, this logistic mapping can be divided into four stages: extinction, stability, period doubling cascade and chaos.

To start, we set µ = 0.9, which is less than 1, Figure 1 shows that the trajectories are attracted to 0. In other words, in the first stage the system is going to “die” or extinction.

Xₜ₊₁ presents a monotonic increase trend, for 1 < µ < 3 in the second stage (Figure 2 & 3). That is to say; the limiting value increases gradually as µ increases, when the value of µ among one to three and the value of Xₜ₊₁ approaches a stable point finally.
Figure 2: Logistic behavior with \( \mu = 1.5, X_0 = 0.9 \)

Figure 3: Logistic behavior with \( \mu = 2.9, X_0 = 0.9 \)

Figure 4: Logistic behavior with \( \mu = 3.0, X_0 = 0.9 \)
Interestingly, if we increased $\mu$ to 3.0, the orbit does not settle down to a stable point, and the system oscillates between two particular points (as Figure 4 shows). According to Salazar Soares (2008), $X_{t+1}$ running cycle between two points is called a period-2 cycle.
Moreover, for $\mu = 3.5$, $X_t$ oscillate between four specific values (Banks, 2003). In this type of oscillation, the evaluation of $X_{t+1}$ repeating every four iterations are named a period-4 cycle.

In the stage of period doubling cascade (Figure 4 to Figure 5 and 6), it can be seen clearly that, the number of cycles has changed from period-2 cycle to a period-4 cycle then a period-8 cycle. These changes are called period doubling.

With the increasing of $\mu$, the bifurcations come more frequently. If $\mu$ increases from 3.5699 and up (but still below 4.0), the system is no longer running as a cycle (Salazar Soares, 2008). At the final stage (Figure 7), $X_{t+1}$ appears to bounce around in a random fashion, which shows in the Figure 7. This behavior looks like chaos.
Based on the analysis above, the figures show that when \( 1 < \mu < 4 \), the logistic map would possess non-trivial dynamical behavior, which also confirmed the viewpoint of May (1976). However, there are several limitations of using figures to show a range of behavior of the logistic equation. Firstly, these figures only can present the logistic behavior in a vague way. For instance, the exact fixed points are hard to be found within the figure. Secondly, compared with algebraic method, figures are more focused on the description of phenomena instead the analysis. Therefore, algebraic method is needed as a tool to work out the accurate fixed point and the period doubling bifurcation.

### 2.3 The period doubling bifurcation

#### 2.3.1 Logistic map of fixed point

Recall the logistic family of mapping \( \mu \mapsto Q_u \), which can be shown as:

\[
Q_u (X) = \mu X (1 - X) \tag{2}
\]

The graphs with the different value of \( Q_u (\mu = 1, 2, 3, 4) \) of four logistic mapping suggest that the change from one or two fixed points occurs when \( \text{id} \) (identity function, which \( f(X) = X \)) is tangent of \( Q_u \) at the fixed point 0. In order to calculate the fixed point, we considered the equation \( Q_u(X) = X \), where the interval of \( X \) is \([0, 1]\). Thus

\[
\mu x(1 - x) = x \tag{3}
\]
To get a fixed point

\[ X_1 = 0, \quad \text{or} \quad X_2 = \frac{\mu - 1}{\mu} \]

As the logistic mapping \( Q_u \) is a quadratic, the result of fixed point can be used to verify algebraically (as Figure 8 shown). In order to analyze the diverse behavior of \( Q_u \) shown by different \( \mu \), we obtain an equation, which shown as

\[ X_{t+1} = X + \varepsilon Y_{t+1} \quad (4) \]

To get

\[ X + \varepsilon Y_{t+1} = \mu \varepsilon Y_t (1 - X) \quad (5) \]

At the fixed point \( X_1 = 0 \), the equation (3) can be written as

\[ Y_{t+1} = \mu Y_t \quad (6) \]

The slope, \( \mu \), of equation (4) can be used to decide whether a fixed point is an attractor or a repellor, which is the same as the definition of fixed point given by Banks, etc. (2003). According to Banks et al. (2003, pp.81), a fixed point is:

\[ \text{Let } f: I \rightarrow \mathbb{R}, \text{ where } I \text{ is an interval. A fixed point } p \text{ of a mapping is called a fixed point if } |f'(p)| = 1. \text{ Otherwise, it is called an attractor (an attracting fixed point) if } |f'(p)| < 1; \text{ or a repellor (a repelling fixed point), if } |f'(p)| > 1. \]
Figure 8: Four logistic mapping:

\[ Q_1 (\mu = 1), Q_2 (\mu = 2), Q_3 (\mu = 3) \text{ and } Q_4 (\mu = 4) \]

*As \( \mu \) increases from 1 to 4, the value of \( Q_u \) increases from 0 to 1.*
Figure 9: A partial bifurcation diagram for $0 < \mu < 3$

*Partial bifurcation diagram showing the stability of the fixed points in the logistic map

Hence, the fixed point $X = 0$ is attracting, if $0 < \mu < 1$. Otherwise, the fixed point $0$ is repelling if $\mu > 1$. Also, when $\mu = 1$ there is only one fixed point $X = 0$, and it is tangent to the graph of $Q_1$.

At the fixed point $X_2 = \frac{\mu - 1}{\mu}$, take this into equation (3) to get

$$\frac{\mu - 1}{\mu} + \varepsilon Y_{t+1} = \mu \left( \frac{\mu - 1}{\mu} + \varepsilon Y_t \right) \left( 1 - \left( \frac{\mu - 1}{\mu} + \varepsilon Y_t \right) \right)$$

(7)

Then

$$Y_{t+1} = (2 - \mu)Y_t$$

(8)

Moreover, the fixed point $X_2 = \frac{\mu - 1}{\mu}$ is attracting when $1 < \mu < 3$, and repelling if $\mu > 3$. Figure 9 can be used to indicate that the stability of fixed points is attractor in the period of $0 < \mu < 3$. 

19
2.3.2 Logistic map of period-2 point

Based on the graphical approach shown in the previous section, when \( \mu > 3 \), the fixed point \( \mu - 1/\mu \) is repelling and the orbit of \( Q_u \) running a cycle between two points (as the Figure 4 show). In other words, with the increasing of \( \mu \), the initial stable points become unstable and bifurcation appears. Recall that the period-2 points of \( Q_u \) are the fixed points of \( Q_u^2 \). In addition, Figure 10 confirmed that when \( \mu = 3 \), the graph of \( Q_u^2 \) is tangent at a fixed point of id; and when \( \mu > 3 \), the logistic mapping, \( Q_u^2 \), have two extra fixed points. Thus, the fixed points of \( Q_u^2 \) bifurcate at \( \mu = 3 \).

Indeed, \( Q_u^2(X) = Q_u(X_{t+2}) = Q_u(Q_u(X)) \), therefore, there are two of the fixed points of \( Q_u^2 \) that are also fixed points of \( Q_u \) which we have already calculated. Hence, when \( \mu > 3 \) an extra pair of fixed points of the logistic mapping \( Q_u^2 \) calculates as follow:

\[
Q_u^2(X) = X
\]  

(9)

Then

\[
\mu (\mu X(1 - X))(1 - \mu X(1 - X)) = X
\]

(10)

To get

\[
X \left( X - \frac{\mu - 1}{\mu} \right) \left( - \mu^3 X^2 + (\mu^3 + \mu^2) X - (\mu^2 + \mu) \right) = 0
\]

(11)
Figure 10: The logistic mapping $Q_u^2$, for the period-2 points

* For $3 < \mu < 4$ there are two extra fixed points in the mapping $Q_u^2$.

Thus,

$$X_{3,4} = \frac{\mu + 1 \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$$

(In addition to the fixed point $X_1 = 0$ and $X_2 = (\mu - 1) / \mu$ of $Q_u$.)
At the fixed point $X_3 = \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$, equation (2) can be rewritten as:

$$X_t = \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu} + \varepsilon Y_t$$  \hspace{1cm} (12)

Therefore,

$$Q_u (X_{t+2}) = \mu (\mu X_t (1 - X_t))(1 - \mu X_t (1 - X_t))$$  \hspace{1cm} (13)

Written as,

$$X_3 + \varepsilon Y_{t+2} = \mu (\mu (X_3 + \varepsilon Y_t)(1 - (X_3 + \varepsilon Y_t)) \right)$$

* (1 - \mu (X_3 + \varepsilon Y_t)(1 - (X_3 + \varepsilon Y_t)))  \hspace{1cm} (14)

As a result,

$$Y_{t+2} = (-\mu^2 + 2\mu + 4)Y_t$$  \hspace{1cm} (15)

Whilst, take the fixed point $X_4 = \frac{\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$ in the equation (11) obtain the same result as $X_3$ does. Thus, the equation (13) can be used to analyze the stability of fixed points. As mentioned above, $|−\mu^2 + 2\mu + 4| < 1$ shows the fixed points are stable. Hence, these two fixed points of $Q_u^2$ are attracting when $3 < \mu < 1 + \sqrt{6}$, otherwise, $X_3$ and $X_4$ are repelling when $1 + \sqrt{6} < \mu < 4$, which is shown in figure 11.
Figure 11: A partial bifurcation diagram for the logistic map under $0 < \mu < 3.5$

May (1976) finds that bifurcation is one of the primary characteristics of the variables in the logistic map. Bifurcation theory shows a partly stable cycle and multiple periodicities transformation along $X_{t+1}$. That is to say if $\mu$ pass the value $1 + \sqrt{6}$, the points of period-2 become unstable and entering the next cycle, the period-4 cycle State.

In this section, we used both graphic approach and algebra to analyze how the fixed points change with different value of $\mu$. It can be seen that all branches can diverge into two embranchments simultaneously by a particular parameter in the description. More precisely, the different kinds of logistic behavior are completely present within the bifurcation diagram (Figure 12). In the interval of $1 < \mu < 3$, there is
Figure 12: The bifurcation diagram

one fixed point in the logistic map, and \( Q \) is stable. After that, the period-2 points (\( \mu = 3 \)) is followed by the period-4 points (\( \mu = 1 + \sqrt{6} \)) and then period-8 points (\( \mu = 1 + \sqrt{8} \)) ... until in a sufficient amount of bifurcation, this system can be proven to enter the chaotic state.

3. EMH and Capital Asset Pricing Model

In finance, efficient market hypothesis and capital asset pricing model are regarded as a framework and standard financial tools in recent years. However, with the continuous improvement in the field of behavioral finance, an increasing number of financial economists doubt the applicability of the efficient market hypothesis (EMH) and
capital asset pricing model (CAPM) in real financial market. They begin to believe finance as a complicated nonlinear dynamic system in which bifurcation and chaos exist. Therefore, a modified asset pricing model, which base on EMH is needed. This section will provide a basic concept and understanding of EMH and capital asset pricing model.

3.1 The Efficient Market Hypothesis

The assumptions of EMH are that all the investors are rational and independent, the prices are efficient; markets self-stabilizing and there is no excess return to collect (Woolley, 2014). That is to say, in an efficient market; no one could predict future prices by using technical analysis of past asset prices or fundamental analysis of company earnings and asset values to outperform the markets or earn extra returns, except by luck or random process. There are three forms in which the efficient market hypothesis is stated: weak form efficiency, semi-strong form efficiency and strong form efficiency. Each form has a different version to define the meaning of “all available information.” In weak form efficiency, stock prices fully reflect information about past stock prices or returns. This weak form implies that historical prices cannot predict future prices. In terms of semi-strong form efficiency, it asserts that all publicly available information must be
reflected in the prices. This public information is not only historical data, but also includes company earnings and dividend prospects, future prices and risks evaluation of a company and so on. However, prices of assets adjust to relevant released information very rapidly and directly, there is no excess return can be earned by using that information (Ball, 1995). Finally, the strong form efficiency states that all public and private information is incorporated in the current prices, even including some resources available only to company insiders. In this form, even corporate insiders are unable to market abnormal profits. However, one of earlier studies documented by Jaffe (1974), stock prices, rise or fall followed by insider bought or sold shares. In addition, according to the approach of Rozef and Zaman (1988), insider traders are profitable; insider profits are 3% per year after transaction cost. Thus, it does not appear to be consistent with the strong form efficient.

3.1.1 Application of the EMH to Financial Markets

According to the concept of “random walk” which was well known by Malkiel (1973), stock prices in the financial markets are random and independent. It is consistent with the efficient market hypothesis. However, there are some economists believe that the market is partly
predictable. Thus, this part will provide some relevant evidences to discuss how the efficient market hypothesis applies to real financial markets.

The weak form efficiency will test first. Based on the random walk theory, successive price movements should be independent, predicting future price is not always accurate. This view could represent weak efficiency form. To test this weak form efficiency, it can be through measuring the serial correlation between the current stock return and the return on the same stock over the previous period. If there is correlation, the return tends to be followed by a tendency for increase or decrease. Otherwise, if there is no correlation between the current stock return and the return on the same stock over the previous period, this could indicate the random walk hypothesis was correct.

According to the finding of Fama (1965) that using a sample of 30 Dow Jones Industrial stocks, the serial correlation coefficient was too small to cover transaction costs of trading. Both Conrad and Kaul (1988) and Lo and Mackinlay (1988) also find a similar result by examining the return of NYSE stock.

Examining the extra returns from technical analysis also could test the weak form of market efficiency. Technical analysis is based on historical prices and the basis of trading patterns and trends. However,
there are controversial views of technical analysis between earlier researches and recent evidences. While earlier researches assert that the technical analysis cannot generate excess profits, Lo et al. (2000) find that some techniques used by professional analysts or chartists may identify or predict the patterns in share prices, such as "head and shoulder" formation and "double bottoms." However, substantial transaction costs and trading cost by using these relative strength strategies or technical analysts are not likely to allow investors to realize excess returns (Lesmond et al., 2001). Consequently, the findings are consistent with weak form efficiency.

On the other hand, the semi strong form of efficiency is involved and has attracted the attention of most economists and practitioners. It mainly focuses on the question of whether using fundamental analysis of public information could improve investment performance. If financial markets are semi strong from efficiency, the prices reflect all publicly available information. Therefore, if the market is efficient, prices would rapidly adjust to new-released information. In other words, once annual report is released to the public, the performance of dividends and profits could quickly reflect the movement of stock prices. If the company is profitable, the stock prices will increase;
otherwise, the opposite reaction will happen. Thus, it does appear to be consistent with the semi strong form efficiency.

3.1.2 Critique of the EMH

Despite most-empirical evidence supporting the efficient market hypothesis, some investors still doubt the efficiency in real financial market. There are some evidences refering to market anomalies, which seem to contradict the efficient market hypothesis, such as under-reaction and over-reaction to new information and low volatility effect.

In the weak form of the efficient market hypothesis, investors are quick to react to new information. However, there are some researches presenting contradictory evidence in some circumstances, like under-reaction and over-reaction to new information. Some limitations can threaten the efficient market hypothesis. Bernard and Thomas (1989) assert that cumulative abnormal returns continue drift up or down to positive or negative earnings for up to 60 days after the announcement. This lagged effect could offer a chance to buy and sell stocks after information released in order to beat the market. In terms of an over-reaction phenomenon, which means investors are actually overacting after purchase. Therefore, both under-reaction and over-
reaction involve the measurement of risk and could explain the effect of price or return momentum.

The low volatility effect presents that portfolios of low volatility stocks could generate higher risk adjusted returns than portfolios of high volatility stocks (Michael and Robert, 1992). The low volatility effect contradicts what the Capital Asset Pricing Model (CAPM) would predict the relationship between risk and return that has been considered as an anomaly. Moreover, more recent studies confirm that this low volatility anomaly have existed about forty years since its initial discovery. This low-volatility anomaly offers potential opportunities for investors, and well supported by behavioral finance and structural characteristics of the marketplace that appear very likely to consistent (Baker and Haugen, 1991). From this phenomenon, investors could consider an allocation to low-risk stocks in order to enjoy the benefit of equity return compounding at a higher rate.

3.2 The asset pricing model

3.2.1 Derivation of the Capital Asset Pricing Model

The capital asset pricing model (CAPM) is a model for pricing a portfolio, and CAPM rests on an important principle in finance, which is called the mean-variance principle (Markowitz, 1952, 1959). The
Markowitz’ model assumes that all the investors are the risk aversion. Moreover, in this model the investors are preferred to gain the portfolio with lower risk and higher expected return, which achieved by focus on the mean and variance of their return.

The expected return of a portfolio in a two-asset is shown as:

$$E(R_p) = \omega E(R_1) + (1 - \omega) E(R_2)$$  \hspace{1cm} (16)$$

Moreover, the variance of the portfolio is

$$VAR(R_p) = \sigma^2(R_p) = \omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2 + 2\omega(1 - \omega)\rho \sigma_1 \sigma_2$$  \hspace{1cm} (17)$$

where proportion of wealth allocation to each asset ($\omega_i$) is $\sum_{i=1}^{N} \omega_i = 1$, and for all $\omega_i$ are non-negative. The correlation coefficient $\rho$ is:

$$\rho = \frac{COV(R_1,R_2)}{\sigma_1 \sigma_2}.$$  

Also, the expected return on N-asset portfolio is:

$$E(R_p) = \sum_{i=1}^{N} \omega_i E(R_i)$$  \hspace{1cm} (18)$$

The variance:

$$VAR(R_p) = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N-1} \omega_i \omega_j COV(R_i,R_j)$$  \hspace{1cm} (19)$$

CAPM is about equilibrium pricing for risky assets in the market and built upon the theory of portfolio selection and diversification. Based on the fundamental equation of CAPM, it can be described as the relationship between the expected return and risk. Under this equation, we begin with three main assumptions. Firstly, investors are
risk-averse. Secondly, investors are price takers and have homogeneous expectation of asset return. That is to say, all the investors have same investment opportunities. Thirdly, the financial market is perfect, which means there are no taxes, no translation cost as well as the CAPM is based on the EMH, that information is costless and available to everyone. These assumptions are idealized and highly simplified, and basic form of CAPM is needed.

In equilibrium, if all these assumptions are satisfied, the expected return of the $i^{th}$ security is that:

$$ ER_i = R_f + (ER_m - R_f) \frac{\sigma_{im}}{\sigma_m} $$

The general idea behind CAPM is that Expected return = risk-free rate + the expected risk premium of investment. In this formula, one of the main purposes for the risk-free rate ($R_f$) is to offset the part of risks that the investors placing money in the investment. The other half of the CAPM formula, the risk premium $(ER_m - R_f) \frac{\sigma_{im}}{\sigma_m}$, represents the amount of compensation the investor needs for taking on additional risk. As mentioned before, the derivation of CAPM is based on the mean-variance principle.

Thus, we assume that the investor hold $\alpha$% in risky asset $i$ and $(1 - \alpha)$ in the market portfolio. The expected return of the portfolio is
\[ E(R_p) = aE(R_i) + (1 - a)ER_m \]  \hspace{1cm} (21)

Variance of portfolio is
\[ \text{VAR} (R_p) = a^2\sigma_i^2 + (1 - a)^2\sigma_m^2 + 2a(1 - a)\sigma_{im} \]  \hspace{1cm} (22)

In the equation (21) and (22), the risk return tradeoff of portfolio can be determined as
\[ \frac{\partial E(R_p) / \partial a}{\partial \sigma(R_p) / \partial a} \]  \hspace{1cm} (23)

The market equilibrium\(^1\) follows only if when \( a = 0 \) in the equation (22). Therefore, the result of equation (23) is
\[ \left. \frac{\partial E(R_p) / \partial a}{\partial \sigma(R_p) / \partial a} \right|_{a=0} = \frac{ER_1 - ER_m}{(\sigma_{im} - \sigma_m^2) / \sigma_m} \]  \hspace{1cm} (24)

Due to the slope of formula of capital market line\(^2\), \( \frac{E(R_m) - R_f}{\sigma_m} \) is based on the market equilibrium, which the same situation as the result of the equation (24) shows.

Thus
\[ \frac{E(R_m) - R_f}{\sigma_m} = \frac{ER_1 - ER_m}{(\sigma_{im} - \sigma_m^2) / \sigma_m} \]  \hspace{1cm} (25)

The result of equation (25) is the same as the equation (20)

\(^1\) **Market equilibrium** is the market state where the supply is equal to the demand in the market (Samuelson, 1983). Moreover, in the equilibrium of market, the price will not change unless the supply or demand changed by an external factor.

\(^2\) **Capital market line (CML)** can be defined as that “if all investors have homogenous beliefs about risk and return, then with riskless borrowing and lending they all have the same linear efficiency frontier” (Markose, 2014). The equation of CML is: \( E(R) = R_f + \left( \frac{E(R_m) - R_f}{\sigma_m} \right) \sigma \)
Let \( \beta_i = \frac{\sigma_{im}}{\sigma_m^2} \). Therefore, a basic formula of CAPM can be written as

\[
ER_i = R_f + \left(ER_m - R_f\right)\frac{\sigma_{im}}{\sigma_m^2}
\]

(20)

\( \beta_i \) represents the relative risk of asset \( i \) (\( \sigma_{im} \)) to the market risk (\( \sigma_m^2 \)) (Krause, 2001).

3.2.2 Critical analysis about CAPM

Based on the derivation of CAPM formulas above, CAPM is provided with a powerfully simple logic that has profound implications for asset pricing and investor behavior. For example, the beta in CAPM is easy to satisfy the requirement of determining expect return by using any risk measure (Perold, 2004). More specifically, if there are two assets, \( \beta_A = 0.7 \) and \( \beta_B = 1.5 \), then the market beta of a 50/50 portfolio is 1.1.

Secondly, the expected return in the CAPM does not depend on its future cash flow. That is to say, under this model, it is not necessary to analyze the financial statement complicatedly. Compare with to explain the expected future cash flows of the firm, only to find the beta in a specific company is much easier.

However, with the work of Krause (2001), Fama and French (2003) and Perold (2004), several limitations are found in the model. Firstly,
the assumptions underlying the CAPM are very strict and take the absence of transaction costs as an example. CAPM assumes trading is costless. But, many investments, such as acquiring a small business involve significant transaction costs. For another example that the CAPM assumes investors have the same beliefs about expect returns and risk of available investments. However, there is massive trading of assets by investors with different expectations and also investors have different risk preferences. Secondly, from the fundamental equation of CAPM, the expected returns can be explained by only one single variable. As mentioned before, it is much easier to use beta to calculate the expected return. However, in the real financial market, other reasonable factors may have an impact on the expected return as well, such as arbitrage pricing theory. Thirdly, according to Roll (1977), if using the market proxy to test the CAPM, the result of the market portfolio being unobservable and the CAPM is not testable. Also, some of the empirical results show that there is no a significant relationship between CAPM and observed returns. Moreover, in the year 2003, Fama and French confirmed that the empirical record of CAPM is weak. However, the practical result is incomplete that if the market proxies, which are used in empirical tests only. As a result, the CAPM has been modified and generated a variety of useful extensions. For the
next section, we are going to analyze the asset pricing model with heterogeneous beliefs.

4. Asset pricing model with the heterogeneous beliefs and bifurcation

4.1 The model

The asset pricing model with the heterogeneous beliefs is one of the best-known extensions that are based on the behavior finance. As mentioned before, all the readers are rational in a perfectly rational EMH world, which is the same the assumption of CAPM whereas traders are different in real financial markets, especially with respect to return and future price. Moreover, the evidences of low volatility effect and efficient market hypothesis are contradictory. Based on the study of Kaizoji (2003), a possible explanation for this phenomenon is that the asset price fluctuations are caused by the interaction between heterogeneous traders, which have different trading strategies and expectations of future prices. Ricchiuti and Maimzada’s (2007) confirmed that the complex dynamics of price fluctuations are related to heterogeneous agent model. That is to say, asset pricing model with
the heterogeneous beliefs, to some extents, can interpret the market anomalies of low volatility effect.

In financial markets, the role of the heterogeneous beliefs can be represented as the different traders have the different expectations about future prices. In the majority of articles of heterogeneous agent model there are two typical types of investors, fundamentalists and chartists. Fundamentalists, as rational “smart money traders” (Brock and Hommes, 1998), believed the asset prices are fully compliance with efficient market hypothesis (EMH) fundamental value that the present discounted value of future dividends is given. On the other hand, chartists (or technical analysts), as “noise trader” (Brock and Hommes, 1998), considered that fundamentals do not completely determine pricing of an asset. The asset prices can be predicted by technical trading rules and observed the trend of past price. In this section, we based on the researches of Hommes and Brock (1998), Riccuuti and Naimzada (2009), Kaizoji (2003) as well as the basic CAPM model, derive and analyze the heterogeneous agent model.

Notations:

The notations in the following are used in the model with heterogeneous beliefs.
As mentioned before, we assume that there are two independent types of investors: fundamentalists and chartists. Within this model, we are going to focus on the risky asset. Due to the different trading strategies of two groups of traders, there are different present of price, which might have an impact on the total amount of traders.

The demand for the risky asset of fundamentalist is based on the difference between fundamental price and the asset price. That is to
say, if the asset price \( (P_t) \) is less than fundamental price \( (P^*) \), (or called undervalued) the fundamentalist might buy this risky asset. Otherwise, if \( P^* - P_t < 0 \), which means the asset is overvalued, the fundamentalist will try to sell the risky asset. Thus, the equation of fundamentalists’ excess demand shows as:

\[
x_t^f = \exp\left(\alpha (P^* - P_t)\right) - 1
\]

where \( \alpha > 0 \);

The fundamentalists are based on the fundamental price. While, chartists are more focus on the expected price. The expected price of chartists at period \( t + 1 \) is based on the expected price at period \( t \) plus the premium adjustment at period \( t \). Therefore;

\[
P_{t+1}^e = P_t^e + \mu (P_t - P_t^e)
\]

where, the correction coefficient \( \mu \in [0, 1] \);

Thus, the chartists’ excess demand of risky asset is given as:

\[
x_t^c = \exp\left(\beta (P_{t+1}^e - P_t)\right) - 1
\]

where \( \beta > 0 \);

From equation (28), it can be seen that the decisions of whether to buy the risky asset for chartists only related to their anticipation about the price in the future. The chartists will buy (sell) the risky asset when it seems that the price will rise (fall) within the next period.
We assume that in the financial market, the market maker can mediate the trading. The market maker applies the following rule:

\[ P_{t+1} = P_t + \theta N[(kx_t^c + (1 - k)x_t^f] \]  \hspace{1cm} (30)

More specifically, if the excess demand is positive in period \( t \), the price will increase in the next period \((t + 1)\) by market maker. Also, the market maker reduces the price for the next period, when the excess demand is negative in period \( t \). As above, the adjustment of price between period \( t \) and next period is that:

\[ P_{t+1} - P_t = \theta N[(kx_t^c + (1 - k)x_t^f] \]  \hspace{1cm} (31)

Moreover, take previous equation (27) and (29) into (31) shows:

\[ P_{t+1} - P_t = \theta N[(1 - k)(\exp(\alpha(P^* - P_t)) - 1) \]

\[ + k(\exp(\beta(P_{t+1}^e - P_t)) - 1)] \]  \hspace{1cm} (32)

### 4.2 Numerical analysis in the model

Based on the description of model, the price dynamical system can be obtained:

\[ P_{t+1} - P_t = \theta N[(1 - k)(\exp(\alpha(P^* - P_t)) - 1) \]

\[ + k(\exp(\beta(1 - \mu)(P_{t+1}^e - P_t)) - 1)] \]

\[ P_{t+1}^e - P_t^e = \mu(P_t^e - P_t) \]  \hspace{1cm} (33)
In order to examine changes that occur in the price dynamic system as the number of active traders \((N)\) varies; we will use both graphical approach and algebraic method in this section. As in the previous section, graphical approach will be used to provide an overview relationship between the number of traders and price fluctuation, which is defined as the price increment \(r_t = P_{t+1} - P_t\) (Kaizoji, 2004). Then, using algebra to confirm the results. We assumed the parameters in this model are restricted as: \(\alpha = 3, \beta = 1, k = 0.5, \mu = 0.5, \theta = 0.001\). Thus, the effect of the different number of traders on the price dynamics can be indicated as follow.

Figure 13: The behavior of dynamic price with “\(\theta N = 0.99\)”
Figure 14: The behavior of dynamic price with \( \theta_N = 1.28 \)

Figure 15: The behavior of dynamic price with \( \theta_N = 1.65 \)

Figure 16: The behavior of dynamic price with \( \theta_N = 2.00 \)
These four figures (Fig. 13-16) represent the four stages respectively. Starting from a small number of $\theta N$, the dynamic prices show a monotonic increase trend (Figure 13), to an oscillation between two specific values (when $\theta N = 1.28$) and a bifurcation occurs in the Figure 14. Moreover, for $\theta N = 1.65$ (Figure 15), the price fluctuation to an oscillation between four specific values. As the value of “$\theta N$” is increased, the bifurcations come faster and faster, until in the final stage (Figure 16), which shows a chaotic behavior.

Recall that the same substitution method will be used to analyze the bifurcations in this system, which shows the impact of increases in the total number of traders $N$ on the price fluctuation. Firstly, there is a unique equilibrium with $P_t = P^* = P^e_t$ in the equation (33), in other words, the $P^*$ is regarded as the fixed point in the price dynamical system.

\[
P_t = P^* + \varepsilon Y_t
\]
\[
P^e_t = P^* + \varepsilon Y^e_t
\]  \hspace{1cm} (34)

Therefore, take equation (34) into the price dynamic system (33) that

\[
\varepsilon (Y_{t+1}^e - Y_t^e) = \theta N [(1 - k)(\exp(-\alpha \varepsilon Y_t))
+ k (\exp(\beta (1 - \mu) \varepsilon (Y_t^e - Y_t)) - 1]
\]
\[
Y_{t+1}^e - Y_t^e = \mu (Y_t - Y_t^e)
\]  \hspace{1cm} (35)
Based on the Taylor’s theorem, equation (35) can be written as:

\[
(Y_{t+1} - Y_t) = \theta N[(\alpha(k - 1) + \beta k(\mu - 1))Y_t + (\beta k(1 - \mu))Y^e_t]
\]

\[
Y^e_{t+1} - Y^e_t = \mu(Y_t - Y^e_t)
\]  

(36)

Elementary computations show that, a sufficient condition for the local stability of the fixed point \(P^*\) in the equation (33)

\[
N < \frac{2(2-\mu)}{\theta (\alpha(2-\mu)(1- k)+2\beta (1-\mu)k)}
\]

(37)

From equation (37) it follows that, with the different number of traders the price dynamic system exhibiting a range of behavior. In other words, if the value of \(\theta N\) is less than 1.09, the behavior of dynamic price shows stability, otherwise the fixed point \(P^*\) is repelling. In other words, the price converges to the fundamental price, when the numbers of traders are less than 1090 (1.09/0.001). On the other hand, the price having bifurcation occur and the price fluctuate significantly when the number of traders exceeds about the 1090.
Figure 17: The bifurcation diagram

* Figure 17 is the bifurcation diagram of price dynamics system, which shows a bifurcation diagram of price increments with 0N as the bifurcation parameter, where $\alpha = 3, \beta = 1, k = 0.5, \mu = 0.5, \theta = 0.001$.

The bifurcation diagram of price dynamic can be used to confirm the result of using algebra method. Therefore, in the Figure 17, it can be seen that the fluctuation of price ($r_t$) with the number of traders ($N$) under the set of parameters shows bifurcations in this system. Based on the analysis above, the orbit of dynamic price does not settle down to the zero-axis, and the price dynamic system oscillates between two particular points. Moreover, if $N$ further increased around 1800, the dynamic system enters the period-4. When $2000 < N < 4000$, as the bifurcation diagram shows, the price increment ($r_t$) enters chaotic state.
5. Conclusion

This paper gives a detailed mathematical derivation of the fixed points and period doubling bifurcation. Moreover, through critical analysis of two basic financial principles, efficient market hypothesis (EMH) and capital asset price model (CAPM), a new model, which is more appropriate to the market, is needed. Therefore, the heterogeneous agent model as a simple model of financial market with two different heterogeneous beliefs: fundamentalists and chartists, is used in this paper. We use the substitution method to find the bifurcation in the heterogeneous agent model. The experiment result was that the dynamic behavior of price in the heterogeneous agent model is related to the total number of active traders. That is to say; the price converges to the fundamental price (when $N<1090$), and as the number of traders exceeds 1090, the chaos of price fluctuation occur through the period doubling cascade.

However, due to the limited researches and sources, there are several limitations and challenges of this paper. Firstly, according to (Kaizoji, 2004), the intermittent chaos of price fluctuation can be observed in the heterogeneous agent model. Based on the work of Kaizoji (2004), if

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3 **Intermittent chaos** is “the weak turbulent state in which the steady motion or the periodic motion is abruptly distributed by the random bursts” (see Berge et al., Order within Chaos, John Wiley and Sons, New York, 1984).
the number of traders (N) is future increased, like when N exceed 4500, the dynamic prices system will re-enter the stage from chaos to bifurcation, and the price increment $r_i$ becomes once again chaotic. However, in this paper, we only focus on the period doubling bifurcation that appeared in the first time. Secondly, the calculation of the equation in the heterogeneous agent model is deficient and it is complicated to verify the period-2 points of heterogeneous agent model by using mathematical derivation. Finally, the types of traders are determined (fundamentalist or chartist) as well as the types of buying strategies are not considered in the asset pricing model with heterogeneous beliefs.

To sum up, based on a basic understanding of the fundamental background and mathematical derivation in the financial area, the heterogeneous agent model present a range of dynamic behavior of prices when the total number of active traders increases. However, due to the limited sources there are some shortcomings are hard to analyze and it may be considered as a significant area of future studies.
Appendix A

Code of the equation (37)

\[
(Y_{t+1} - Y_t) = \theta N[(\alpha (k - 1) + \beta k (\mu - 1))Y_t + (\beta k (1 - \mu))Y_t^e]
\]

\[
Y_t^e - Y_t^e = \mu(Y_t - Y_t^e)
\]  (36)

The equation (36) can be written in a matrix, which is

\[
\begin{pmatrix}
Y_{t+1} - Y_t \\
Y_t^e - Y_t^e
\end{pmatrix} = \begin{pmatrix}
\theta N(\alpha (K - 1) + \beta K (\mu - 1)) & \beta K (1 - \mu) \\
\mu & -\mu
\end{pmatrix} \begin{pmatrix}
Y_t \\
Y_t^e
\end{pmatrix}
\]

(A.1)

Using the eigenvalues in this matrix, therefore, the eigenvalue equation for the matrix A is

\[(A - \lambda I)\nu = 0\]  (A.2)

Thus

\[
\det(A - \lambda I) = \det \left[ \begin{array}{cc}
\theta N(\alpha (K - 1) + \beta K (\mu - 1)) & \beta K (1 - \mu) \\
\mu & -\mu
\end{array} \right] - \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(A.3)

\[\det(A - \lambda I) = \det \left[ \begin{array}{cc}
\theta N(\alpha (K - 1) + \beta K (\mu - 1)) & -\lambda \beta K (1 - \mu) \\
\mu & -\mu - \lambda
\end{array} \right] \]  (A.4)

Due to the definition of fixed point by Bank et al. (2003),

Let \(|\lambda| < 1\), therefore,

\[N < \frac{2(2-\mu)}{\theta|\alpha(2-\mu)(1-k)+2\beta(1-\mu)k|} \]  (A.5)
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