Solutions of a PT-symmetric Dimer with Constant Gain-loss

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School of Mathematical Sciences
University of Nottingham

John Pickton

Supervisor: Dr Hadi Susanto
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Abstract

Solutions to the PT-symmetric dimer with constant gain-loss are found. It is shown that the well-known three-dimensional equations can be reduced to one pendulum-like equation containing one dependent variable. This is done by finding two constants of motion and by the introduction of a new variable. Qualitative solutions are found by analysing phase planes. A strong link with the geometry of circles is also discussed. This paper might lead the way in finding solutions when the gain-loss coefficient is time-dependent.
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1 Introduction

Due to extensive experimental verifications, quantum mechanics is a widely accepted area of science and studies of the natural world. Being over One-hundred years old, it is built upon a set of four rules known as axioms. Three of these axioms have direct physical interpretations. The energy of a particle must be real and it must also be bounded below. Also, a system of particles must also be unitary, in other words probability conserving. The fourth axiom seems to be different from the other three because it is instead mathematical in its nature. This final rule states that the Hamiltonian of a system must be Hermitian.

It has been conjectured, and explained in [1] and [3], that the axiom for Hermiticity of a quantum system may be replaced by the broader condition of Parity-Time (PT) symmetry. This means that the Hamiltonian, $\hat{H}(\hat{x},\hat{p},t)$, must be invariant under the combination of the parity reflection $\hat{x} \rightarrow -\hat{x}$ and the time reflection $t \rightarrow -t$, $i \rightarrow -i$. It has been shown mathematically that if PT-symmetry holds the system still give a real energy spectra. However, this result must be verified experimentally. In this project we study one such non-Hermitian system; a PT-symmetric dimer/coupler with a Kerr non-linearity and with a gain loss. The measure of the gain loss is given by the coefficient $\gamma$. Due to symmetries of the equations this is take to be non-negative. When $\gamma$ is equal to zero the equations are Hermitian. However, when $\gamma$ is non zero the system is not longer Hermitian. Instead the solutions are PT-symmetric. Any unstable solutions will break this symmetry so understanding these equations mathematically can prove useful when the model is tested experimentally. Optics has been used so that such experiments have been carried out by [5] for the linear equations where results found that there was a critical value of $\gamma$ where solutions had broken PT-symmetry. There are several papers on numerical simulations of the equations but
The aim of the work here is to try to explain solutions of the PT-symmetric dimer. It is known that the equations can be reduced to three variables. This means that solutions could potentially be chaotic. However, it can be shown that this is not the case. Comparisons to the equations with $\gamma = 0$ will be useful in understanding what effect the gain-loss coefficient has on solutions. One paper [6] discusses the integrability of the system. This agrees with the results of this project.

When $\gamma = 0$ a phenomena known as self trapping is observed. Here a solution trajectory oscillates in the with either $|u_1| > |u_2|$ or the other way around. It can be shown that such a trajectory does not exist for $\gamma$ non-zero.

The project begins by familiarising the reader with different forms of the governing equations in section 2. This involves introducing new variables such as the power in the system. In section 3 it is shown that a constant of motion, which we call $c$, can be found. This constant then allows solutions to be categorised depending on what value $\gamma$ takes. This is summaries in figure 1 Conditions for equilibrium points and their positions are discussed in section 4. Methods are developed over sections 5 and 6 to find qualitative solutions. The first of these sections works with $\gamma = 0$, using the reduced difficulty of the problem to develop arguments for analysis. methods are then refined to include $\gamma \neq 0$ where a wider range of solutions is discussed.

In section 7 a different approach is taken to prove integrability of the equations. This involves linking the equations we the geometry of circles. A parameter is then introduced in order to reduce the system to one differential equation. This equation is comparable to that of a pendulum. A tree diagram gives a rough breakdown of the project.

The main results of the project could possibly provide some steps towards
analysing the equations when $\gamma$ is time dependent.

Figure 1: A tree diagram of the structure of the project

\begin{figure}
\centering
\begin{tikzpicture}

\node [rectangle, draw] (y0) {$\gamma = 0$} edge[->] node[right] {$c = 0$} (c0);
\node [rectangle, draw] at (y0-|c0) (c01) {$c \in (0,1)$} edge[->] node[right] {$c = 1$} (c1);
\node [rectangle, draw] at (c01-|c1) (c11) {$c \geq 1$};
\node [rectangle, draw] (y1) at (y0-|c1) {$\gamma \geq 1$};

\node [rectangle, draw] (y11) at (y1-|c01) {$\gamma \in (0,1)$} edge[->] node[right] {$c \in (0,\gamma]$} (c1);
\node [rectangle, draw] at (y11-|c1) (c11) {$c \in (\gamma,1)$} edge[->] node[right] {$c = 1$} (c1);
\node [rectangle, draw] at (c11-|c1) (c111) {$c \geq 1$};
\end{tikzpicture}
\end{figure}
2 Governing Equations

2.1 Equations of a coupled wave guide

The equations modelling a coupled dimer with a Kerr non-linearity are given by:

\[\begin{align*}
    i\dot{u}_1 &= -\kappa u_2 - \delta|u_1|^2u_1 - i\gamma u_1, \\
    i\dot{u}_2 &= -\kappa u_1 - \delta|u_2|^2u_2 + i\gamma u_2.
\end{align*}\]  
(2.1a, 2.1b)

Here; \(\kappa\) is known as the coupling constant, \(\delta\) is a measure of the Kerr non-linearity and \(\gamma\) is called the gain-loss coefficient. The 'dot' represents differentiation with respect to the variable \(t\). These equations are, in general, non-Hermitian but hold the property of PT-symmetry. Solutions have been found when \(\gamma\) is identically zero. As we find solutions to the non-zero \(\gamma\) case, these previously found results provide a good comparison. By the symmetry of the equations it is acceptable to assume that \(\gamma\) is non-negative.

It is not uncommon to find the above equations written in a slightly simpler form:

\[\begin{align*}
    i\dot{u}_1 &= -u_2 - |u_1|^2u_1 - i\gamma u_1, \\
    i\dot{u}_2 &= -u_1 - |u_2|^2u_2 + i\gamma u_2.
\end{align*}\]  
(2.2a, 2.2b)

Although this may seem, at first sight, a less general form of the governing system, it is actually produced by rescaling the variables by:

\[\begin{align*}
    t &\rightarrow \frac{t}{\kappa}, & \gamma &\rightarrow \kappa \gamma, & u_1 &\rightarrow \sqrt{\frac{\kappa}{\delta}}u_1, & u_2 &\rightarrow \sqrt{\frac{\kappa}{\delta}}u_2.
\end{align*}\]  
(2.3)

At this point it is appropriate to talk about what the above equation is actually describing physically. The complex variables \(u_1\) and \(u_2\) are described as wave functions. They do not represent a possible solution for a particle travelling along the waveguide but rather the superposition of many such particles. The
magnitudes $|u_1|$ and $|u_2|$ are representative of the probability of finding a particle in the respective waveguides. In fact, the probabilities are respectively given by; $|u_1|^2/P$ and $|u_2|^2/P$, where $P = |u_1|^2 + |u_2|^2$ is the 'power' in the system. The probability difference between the two waveguides can be measured by a variable; $\Delta = |u_2|^2 - |u_1|^2$. Having briefly explained the magnitudes of $u_1$ and $u_2$ it may be helpful to give some meaning to the argument angles $\phi_1$ and $\phi_2$. Throughout this project we refer to such angles as the 'phase' in the waveguide. Unlike the magnitudes, the phase angles do not have such a direct meaning. However, it is believed that knowledge of these is useful in studying particle collisions in Interferometry, see [8]. This so called phase is undefined when the corresponding magnitude is zero. The instance where either $u_1 = 0$ or $u_2 = 0$ can be described straight from the governing equations and is done so in appendix A. This shows that the non-existence of the phase angle in this moment does not present any problems to the analysis in later sections.

2.2 Equations in polar form

Assume that $u_1$ and $u_2$ can be written in polar form, i.e.;

$$u_1 = |u_1|e^{i\phi_1}, \quad u_2 = |u_2|e^{i\phi_2}.$$  \hfill (2.4)

By substituting into the governing equations, the system can be reduced to three dimensions by introducing the phase difference $\theta = \phi_2 - \phi_1$. This is verified in appendix A, but the result is;

$$\frac{d}{dt}|u_1| = -|u_2|\sin \theta - \gamma |u_1|,$$  \hfill (2.5a)

$$\frac{d}{dt}|u_2| = |u_1|\sin \theta + \gamma |u_2|,$$  \hfill (2.5b)

$$\frac{d\theta}{dt} = \left(|u_2|^2 - |u_1|^2\right) \left(1 - \frac{\cos \theta}{|u_1||u_2|}\right).$$  \hfill (2.5c)
2.3 Equations in terms of $P$, $\Delta$ and $\theta$

From the equations in the previous section, we can also express the system in terms of the variables $P$ and $\Delta$. The definitions of these were;

$$ P = |u_1|^2 + |u_2|^2, \quad \Delta = |u_2|^2 - |u_1|^2. \quad (2.6) $$

Then by making use of the product rule of differentiation, the equations are given;

$$ \frac{dP}{dt} = 2\gamma \Delta, \quad (2.7a) $$
$$ \frac{d\Delta}{dt} = 2\gamma P + 2\sqrt{P^2 - \Delta^2} \sin \theta, \quad (2.7b) $$
$$ \frac{d\theta}{dt} = \Delta \left(1 - \frac{2 \cos \theta}{\sqrt{P^2 - \Delta^2}}\right). \quad (2.7c) $$

Appendix A gives the full derivation of these equations.
The First Constant of Motion

The dimensions of the problem can be reduced from three to two by finding a constant of motion. This happens to be very useful as solutions can then be drawn out in phase planes. Moreover, I proves the in-existence of chaotic solutions. Full details of calculations a given in the appendix B, however several steps will be shown here, as to give some immediate insight as to where resulting equations come from. Firstly, through use of the product rule for differentiation in equations 2.5a, 2.5b and 2.5c we can work out that;

\[ \frac{d}{dt}(|u_1||u_2|\cos \theta) = |u_1||u_2| (|u_1|^2 - |u_2|^2) \sin \theta. \]  

(3.1)

In a similar manner it can also be seen that;

\[ \frac{d}{dt}(|u_1|^2|u_2|^2) = 2|u_1||u_2| (|u_1|^2 - |u_2|^2) \sin \theta. \]  

(3.2)

Hence, these two expressions are only different by a factor of two. By integrating with respect to \( t \) it soon follows that;

\[ |u_1||u_2| (|u_1||u_2| - 2 \cos \theta) = c^2 - 1 = \text{constant}. \]  

(3.3)

With no loss of generality this constant has been set equal to \( c^2 - 1 \). This choice of \( c \) was originally introduce to simplify following expressions. However, in a later section we see that there is a deeper reason why this choice is useful. Notice that equation 3.3 is quadratic in \( |u_1||u_2| \). It can therefore be written;

\[ |u_1||u_2| = \cos \theta \pm \sqrt{c^2 - \sin^2 \theta} \]  

(3.4)

Due to the conditions that \( |u_1||u_2| \) must be real and positive; it follows that \( c \) is real and non-negative. There are four separate cases for \( c \) that have different implications of what values \( \theta \) can take. For \( c = 0 \);

\[ \theta = 0 \quad \text{and} \quad |u_1||u_2| = 1. \]  

(3.5)
For $0 < c < 1$;

$$\theta \in [-\arcsin c, \arcsin c] \quad \text{and} \quad |u_1||u_2| = \cos \theta \pm \sqrt{c^2 - \sin^2 \theta}. \quad (3.6)$$

For $c = 1$;

$$\theta \in (-\pi/2, \pi/2) \quad \text{and} \quad |u_1||u_2| = 2 \cos \theta. \quad (3.7)$$

For $c > 1$;

$$\theta \in [-\pi, \pi) \quad \text{and} \quad |u_1||u_2| = \cos \theta + \sqrt{c^2 - \sin^2 \theta}. \quad (3.8)$$

We can find the $c$ value of any trajectory by the initial values. The qualitative behaviour of the trajectory then depends on which of the four cases above that $c$ falls into. These different cases are described by figure 2. A few more things can be said about each of the four cases.

### 3.1 $c = 0$

Explicit solutions can be found for trajectories with $c = 0$. By substituting into equation 2.5a to get;

$$\frac{d}{dt}|u_1| = -\gamma |u_1|. \quad (3.9)$$

The equation is easily solved to give;

$$|u_1| = Ae^{-\gamma t} \quad \text{and} \quad |u_2| = \frac{1}{A}e^{\gamma t}. \quad (3.10)$$

Here $A$ is an arbitrary constant to be determined by the initial conditions. Already we can see major differences between solutions with $\gamma = 0$ and $\gamma > 0$. In the former, $c = 0$ represents a line of equilibrium points whereas in the latter, $|u_1|$ decays exponentially and $|u_2|$ grows exponentially. This shows that unstable solutions exist for any $\gamma \neq 0$. 14
Figure 2: Diagram showing the relationship between $|u_1||u_2|$ and $\theta$ for three different values of the constant $c$. The (purple) line for $c > 1$ shows how the phase angle $\theta$ is unbounded. The (blue) line for $c = 1$ explains why we might define $\theta = \pm \pi/2$ when $|u_1||u_2| = 0$. The (green) line for $0 < c < 1$ demonstrates how one value of $\theta$ can correspond to two values of $|u_1||u_2|$. Here it can be seen that $\theta$ is indeed bounded.
3.2 $0 < c < 1$

3.2.1 The upper and lower regions

When $0 < c < 1$ both the $+$ and $-$ solutions of equation 3.4 exist. When talking about trajectories with such a $c$-value we must be clear about what solution $|u_1||u_2|$ is satisfying. Denote the trajectory being in the upper region as meaning:

$$|u_1||u_2| = \cos \theta + \sqrt{c^2 - \sin^2 \theta}.$$  \hfill (3.11)

Similarly say that the trajectory is in the lower region when:

$$|u_1||u_2| = \cos \theta - \sqrt{c^2 - \sin^2 \theta}.$$  \hfill (3.12)

A sketch of these regions is shown in figure 3.

![Diagram](https://example.com/diagram.png)

**Figure 3:** This figure shows how a trajectory with $c \in (0, 1)$ be said to be in one of two regions. For a given $\theta \in (-\arcsin c, \arcsin c)$; $|u_1||u_2|$ can take one of two values. The upper region signifies the largest of these two values while the lower region corresponds to the lesser.
3.2.2 Behaviour at the boundaries of $\theta$

At this point find out what the trajectory does at the boundaries of $\theta$. It is seen that this is the point where the upper and lower region meet. Given that $|u_1||u_2|$ is assumed to be continuous, it follows that a trajectory can switch between the two regions at the boundaries. What region a trajectory moves into at the boundaries can be analysed by looking at the rate of change of $|u_1||u_2|$ with respect to $t$. At $\theta = -\arcsin c$;

$$\frac{d}{dt} (|u_1||u_2|) = c\Delta.$$  \hfill (3.13)

At $\theta = \arcsin c$;

$$\frac{d}{dt} (|u_1||u_2|) = -c\Delta.$$  \hfill (3.14)

Therefore if $\Delta > 0$ a trajectory switches the upper region at $\theta = -\arcsin c$ and switches to the lower region at $\theta = \arcsin c$. If $\Delta < 0$ the opposite happens. This becomes more apparent in section 5. Figure 4 gives the directions of trajectories in three different regions.

3.3 $c = 1$

The case where $c = 1$ contains all solutions such that $|u_1||u_2| = 0$ at any time $t$. At this point the phase difference $\theta$ is undefined, however we will see that it at such a point it proves useful to define $\theta = \pm \pi/2$. This follows from substituting $c = 1$ into equation 3.4 to get;

$$|u_1||u_2| = 2 \cos \theta.$$  \hfill (3.15)

Remember that technically the value of the phase difference is bounded between

$-\pi/2 < \theta < \pi/2$. 

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Figure 4: This diagram illustrates how the sign of $\Delta$ determines which direction trajectories with different values of $c$ travel around the curves of $|u_1||u_2|$ against $\theta$. In this figure $\Delta$ is assumed positive. For $c \in (0,1)$; trajectories transverse in a 'clockwise' fashion, entering the upper region at $\theta = -\arcsin c$ and the lower region at $\theta = \arcsin c$. The graph corresponding to $\Delta$ negative is obtained by reversing the direction of the arrows.
3.4  \( c > 1 \)

The final case to consider is that of \( c \) greater than unity. This case gives no physical boundary on the phase difference \( \theta \). However, the \( - \) solution in equation 3.4 cannot exist so it follows that \(|u_1||u_2| = \cos \theta + \sqrt{c^2 - \sin^2 \theta}|.\)
4 Equilibrium Points

Like any analysis of a system of differential equations it is useful to find out if there exist any equilibrium points. If equilibrium points would happen to exist it then proves useful to find out the location, and then the nature of these points. In this section we do the first two of these things. Beginning with finding the necessary conditions for equilibrium points to exist we show that there is a threshold of PT-symmetry breaking for a certain value of the gain loss coefficient, $\gamma$. It is important to compare the main differences between $\gamma = 0$ and $\gamma > 0$ so we will discuss these separately. Secondly, we find out how the value of $c$ affects the position of the equilibrium points.

4.1 Equilibrium points when $\gamma = 0$

4.1.1 Conditions for equilibrium points

From equations 2.5a and 2.5b it follows that an equilibrium point must have $\theta = 0$ or $\theta = -\pi$. By substituting this into equation 2.5c gives the following possible conditions for an equilibrium point;

$$|u_1| = |u_2| \quad \text{and} \quad \theta = 0,$$

$$|u_1| = |u_2| \quad \text{and} \quad \theta = -\pi,$$

$$|u_1||u_2| = 1 \quad \text{and} \quad \theta = 0.$$  

The last of these three possible conditions corresponds to trajectories with $c = 0$.

4.1.2 Positions of equilibrium points for $c = 0$

For this value of $c$ trajectories satisfy $|u_1||u_2| = 1$ and $\theta = 0$. In other words all of these points are in fact equilibrium points. The difference between these
equilibrium points and any others is that they do not require $\Delta = 0$. Because of this, trajectories can form a phenomena known as self-trapping (REFERENCE).

4.1.3 Positions of equilibrium points for $0 < c < 1$

We have seen that for this case of $c \in (0, 1)$, $\theta$ is bounded between $-\arcsin c$ and $\arcsin c$. This means that an equilibrium point at $\theta = -\pi$ is not feasible. Thus any equilibrium points must have $\theta = 0$. By using the fact that $\Delta = 0$ here we can find the power at the equilibrium points by use of $P = 2 \cos \theta \pm 2 \sqrt{c^2 - \sin^2 \theta}$ to be $P = 2 + 2c$ and $P = 2 - 2c$. Thus there are two equilibrium points;

- $\theta = 0$ and $P = 2 + 2c$
- $\theta = 0$ and $P = 2 - 2c$

4.1.4 Positions of equilibrium points for $c = 1$

We already know that this case of $c$ contains the trivial equilibrium point with $P = 0$. Given the boundaries imposed on the phase difference here it follows that the other equilibrium point exists at $\theta = 0$. We can use $\Delta = 0$ to say $P = 4 \cos \theta$. Thus;

- $\theta = 0$ and $P = 4$
- $(\theta = \pm \pi/2)$ and $P = 0$

are the two equilibrium points when $c = 1$. Note that for the trivial equilibrium point; we have chosen to define $\theta$ even though it is technically undefined.

4.1.5 Positions of equilibrium points for $c > 1$

$\theta$ is not physically bounded in this case to there can be an equilibrium point at $\theta = -\pi$ as well as at $\theta = 0$. This can be used to say that the two equilibrium points are given by;

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\[ \theta = 0 \text{ and } P = 2 + 2c \]

\[ \theta = -\pi \text{ and } P = 2c - 2 \]

A sketch of a bifurcation diagram for the power of equilibrium points against \( c \) is given by figure 5.

Figure 5: This sketch shows a bifurcations diagram of the power for equilibrium points against the constant \( c \). Here \( \gamma = 0 \).

4.2 Equilibrium points when \( 0 < \gamma < 1 \)

4.2.1 Conditions for equilibrium points

By analysing the equations 2.7a, 2.7b and 2.7c we can find conditions for equilibrium points when \( \gamma \neq 0 \). Any non-trivial equilibrium points must satisfy:

\[ \Delta = 0 \quad \text{and} \quad \sin \theta + \gamma = 0. \quad (4.2) \]

As seen in figure 6. Therefore, it is easily seen that no such equilibrium points exist for \( \gamma > 1 \). This is a stability threshold similar to that found in the linear
case [5]. Notice that if $c < \gamma$ then the allowed range of $\theta$ does not include the equilibrium points. Hence no equilibrium points exist in this case. Thus there must be a bifurcation when $c = \gamma$ from no equilibrium points to two. It can be seen that no non-trivial equilibrium points can be found when $\Delta$, unlike when $\gamma$. Therefore, there are no self-trapped states when $\gamma > 0$.

\[
-\pi + \arcsin \gamma - \arcsin \gamma - \gamma \sin \theta
\]

Figure 6: This diagram shows the $\theta$ values of possible non-trivial equilibrium points. When $\gamma \in (0, 1)$; such points can be found at either $\theta = -\arcsin \gamma$ or $\theta = -\pi + \arcsin \gamma$. From the graph it is clear that no non-trivial equilibrium can exist.

4.2.2 Positions of equilibrium points for $c = \gamma$

This is the value of $c$ for which there is a bifurcation from no equilibrium points to two. Because the equilibrium point here is on the boundary for $\theta$ there is only one equilibrium point found at;

- $\theta = -\arcsin \gamma$ and $P = 2\sqrt{1 - \gamma^2}$
4.2.3 Positions of equilibrium points for $\gamma < c < 1$

We have seen that for this case of $c$, $\theta$ is bounded between $-\arcsin c$ and $\arcsin c$. This means that any equilibrium points must be found at $\theta = -\arcsin \gamma$. By using the fact that $\Delta = 0$ here we can find the power at the equilibrium points by use of $P = 2 \cos \theta \pm 2 \sqrt{c^2 - \sin^2 \theta}$. Take note that at the equilibrium points $\cos \theta = \sqrt{1 - \gamma^2}$. Thus for the equilibrium points:

- $\theta = -\arcsin \gamma$ and $P = 2\sqrt{1 - \gamma^2} + 2\sqrt{c^2 - \gamma^2}$
- $\theta = -\arcsin \gamma$ and $P = 2\sqrt{1 - \gamma^2} - 2\sqrt{c^2 - \gamma^2}$

4.2.4 Positions of equilibrium points for $c = 1$

We already know that this case of $c$ contains the trivial equilibrium point with $P = 0$. Given the boundaries imposed on the phase difference here it follows that the other equilibrium point exists at $\theta = -\arcsin \gamma$. We can use $\Delta = 0$ to say $P = 4 \cos \theta$. Thus;

- $\theta = -\arcsin \gamma$ and $P = 4\sqrt{1 - \gamma^2}$

4.2.5 Positions of equilibrium points for $c > 1$

$\theta$ is not physically bounded in this case to there can be an equilibrium point at $\theta = -\pi + \arcsin \gamma$ as well as at $\theta = -\arcsin \gamma$. This can be used to say that equilibrium points exist at;

- $\theta = -\arcsin \gamma$, $P = 2\sqrt{1 - \gamma^2} + 2\sqrt{c^2 - \gamma^2}$
- $\theta = -\pi + \arcsin \gamma$, $P = -2\sqrt{1 - \gamma^2} + 2\sqrt{c^2 - \gamma^2}$

Figure 7 gives a sketch of the bifurcation diagram for the power of equilibrium points against $c$ when $\gamma \in (0, 1)$.
4.3 Equilibrium points when $\gamma = 1$

As we have said before, this value of the gain loss coefficient is the threshold for stability. From equations 4.2 it can be seen that any equilibrium point must have $\theta = -\pi/2$. Thus no non-trivial equilibrium points exist for $c \leq 1$.

4.3.1 Positions of equilibrium points for $c > 1$

The position of these equilibrium points is given by:

- $\theta = -\pi/2$ and $P = 2\sqrt{c^2 - 1}$

Figure 7: This sketch shows a bifurcations diagram of the power for equilibrium points against the constant $c$. Here $\gamma \in (0, 1)$
5 Phase plane analysis for $\gamma = 0$

In this section we introduce methods of analysing solutions of a PT-symmetric dimer with a zero gain-loss coefficient, i.e. $\gamma = 0$. It is recommended that section be read before reading the sections on the analysis of $\gamma > 0$ as this will provide a smoother explanation of some of the methods.

This section is categorised into four subsections. These separately study trajectories with different the cases of the constant $c$; $c = 0$, $c > 1$, $c = 1$ and $c \in (0, 1)$. In the last of these sections the self-trapped state is described.

All of the following sections take advantage of the fact that the power, $P$, remains constant when $\gamma = 0$. This is seen from equation 2.7a.

5.1 $c = 0$

Firstly we cover the most simple class of solutions; those with $c = 0$. In section 3 it was found that $\theta = 0$ and $|u_1||u_2| = 1$ for all $t$. Putting this information into equations 2.5a and 2.5b results in the very simple;

$$\frac{d}{dt}|u_1| = 0, \quad \frac{d}{dt}|u_2| = 0. \quad (5.1)$$

Therefore the family of trajectories with $c = 0$ is a line of equilibrium points.

5.2 $c > 1$

Trajectories with $c$ greater than one are more complex than those with $c$ equal to zero, however they provide a good starting point for developing a method of analysis due to the fact that it is the only case when $\theta$ remains unbounded. Begin by recalling that trajectories in this classification obey the equation;

$$\sqrt{P^2 - \Delta^2} = 2 \cos \theta + 2\sqrt{c^2 - \sin^2 \theta}. \quad (5.2)$$
5.2.1 The power is bounded below

Using the inequality $P \geq \sqrt{\Delta^2 - \Delta^2}$ we can find lower bound of $P$ to be the value of the power when $\Delta = 0$. This will be denoted by $P_{\text{min}}$. Thus;

$$P_{\text{min}} = 2 \cos \theta + 2 \sqrt{c^2 - \sin^2 \theta}.$$  \hfill (5.3)

It corresponds to the minimum value $P$ can take for a given $\theta$ and is shown in figure 8 as a bell-shaped (orange) curve. This curve is a boundary through which trajectories cannot pass below. It should be noted that because this line corresponds to $\Delta = 0$; the two equilibrium points lie on this curve at $\theta = 0$ and $\theta = -\pi$.

5.2.2 Direction of trajectories

The direction that trajectories take depends on what the sign of $\Delta$ is. We have already observed that for $c > 1$; $\theta$ will increase if $\Delta > 0$ and decrease if $\Delta < 0$. When $\Delta = 0$ the variable $\theta$ is stationary. The rate of change of $\Delta$ is given by equation 2.7b. It follows that $\Delta$ is stationary for $\theta = -\pi$ or $\theta = 0$. otherwise $\Delta$ increases if $\theta$ is positive and decreases if $\theta$ is negative. This is consistent with figure 4 also describing directions of trajectories.

5.2.3 Behaviour of trajectories in the $(\theta, P)$-plane

Trajectories in the $(\theta, P)$-plane follow horizontal lines, because $P$ is constant. Having found a lower boundary for $P$, through which trajectories do not pass, it follows that solutions can behave in two different ways depending on what power they have. This divides the phase plane into two separate regions. Before we go deeper into this we note that the lowest possible power of a solution is given by the power at the equilibrium point with $\theta = -\pi$. From the equation 5.3 we find this lowest power to be $P = 2(c - 1)$. Now, the phase plane can be separated.
into; trajectories with \( P > 2(c + 1) \) and \( 2(c - 1) < P < 2(c + 1) \). Solutions in these two regions are explained separately.

Trajectories with \( P > 2(c + 1) \) have a higher power than the maximum value of \( P_{\text{min}} \). This means that they never reach \( \Delta = 0 \) and therefore have either \( \Delta \) positive or \( \Delta \) negative for all values of \( t \). The direction of the trajectories is therefore constant and dependent solely on the sign of \( \Delta \). Figure 8 gives an example of such a trajectory labeled 'Stable 2'.

Trajectories with \( 2(c - 1) < P < 2(c + 1) \), illustrated in figure 8 as 'Stable 1', do reach \( P_{\text{min}} \) at some point. When this happens a trajectory has reached a point where \( \Delta = 0 \) and thus its \( \theta \) value is stationary. The resulting direction of the solution at this instances then depends on the sign of \( d\Delta/dt \) and hence \( \theta \), using equation 2.7b. If \( \theta > 0 \) the trajectory proceeds to move with increasing \( \theta \), whereas if \( \theta < 0 \) it moves with decreasing \( \theta \). The solution will continue to move in such a way until it again reaches \( \Delta = 0 \). Using this information we can deduce that the solution will be periodic with the trajectory oscillating back and forth.

\subsection*{5.2.4 Existence of a separatrix}

We have found two separate regions containing qualitatively different solutions. It was shown that these were categorised by trajectories with power above \( P = 2(c+1) \) and those with power below. However, nothing has been said so far about a possible trajectory with \( P = 2(c+1) \). So what happens to such a trajectory? It is observed that the trajectory will touch \( P_{\text{min}} \) at one point; an equilibrium point. Once this has happened the solution will remain there forever. The direction in which this happens depends on the sign of \( \Delta \) but is irrelevant here. At this point we discuss the possibility of such a solution physically. As we are studying a model it is important to remember that we have discounted the effects of any small perturbation in the physical system. Because of this, it is almost impossible
Figure 8: This figure gives examples of trajectories with $\gamma$ equals zero and $c$ greater than one. It shows two different trajectories, one above the separatrix and one below. The separatrix is given by the (blue) dotted line.

to produce a trajectory with an exact power. Therefore, a trajectory is almost certain to fall into one of the two regions discussed above. When performing our analysis we call this path the separatrix. An sketch of a separatrix is given by the dotted line in illustrated in figure 8.

5.2.5 Trajectories in the $(\theta, \Delta)$-plane

Having found the solutions in the $(\theta, P)$-plane we can compare trajectories in the $(\theta, \Delta)$-plane. An expression for the trajectories is given by rearranging equation 5.2 to get;

$$\Delta^2 = P^2 - 4 \left( \cos \theta + \sqrt{c^2 - \sin^2 \theta} \right)^2,$$

although this isn’t needed to get an idea of the shape trajectories will take. The fig. shows how trajectories in the $(\theta, P)$-plane relate to those in the $(\theta, \Delta)$-plane.
Firstly we see that trajectories with $P > 2(c + 1)$ oscillate their value of $\Delta$, appearing to be pulled towards and then away from the equilibrium point at $\theta = \Delta = 0$. Trajectories with $2(c - 1) < P < 2(c + 1)$ oscillate around the other equilibrium point at $\theta = -\pi$ and $\Delta = 0$ in what can be described as a clockwise direction. Finally we note that the two solutions are separated by the separatrix passing through $\theta = \Delta = 0$. This is all illustrated in figure 9.

![Diagram](image)

Figure 9: This figure shows trajectories with $\gamma$ equals zero and $c$ greater than one corresponding to figure 8.

### 5.3 $c = 1$

After studying trajectories with $c$ greater than one; we find that the analysis of trajectories with $c$ equal to one is not too different. In fact the only major difference is that the phase angle is bounded, $\theta \in (-\pi/2, \pi/2)$. For this reason we can give a more brief overview of the methods used to analyse such solutions.
Like before, recall that trajectories in this classification obey the equation;

$$\sqrt{P^2 - \Delta^2} = 4 \cos \theta.$$  \hfill (5.5)

### 5.3.1 The power is bounded below

We can find lower bound of $P$ to be the value of the power when $\Delta = 0$. Denote this by $P_{\text{min}}$. Thus;

$$P_{\text{min}} = 4 \cos \theta.$$  \hfill (5.6)

This is shown in the diagram of the $(\theta,P)$-plane as a curve (orange) in figure 10. This curve is a boundary through which trajectories cannot pass below. The one non-trivial equilibrium point lies on this curve at $\theta = 0$. The other equilibrium point is trivial i.e. $P = 0$ and is represented at $\theta = -\pi/2$.

### 5.3.2 Direction of trajectories

The direction that trajectories take depends on what the sign of $\Delta$ is and is explained in the same was as for when $c > 1$.

### 5.3.3 Behaviour of trajectories in the $(\theta,P)$-plane

Just as for the $c > 1$ case, trajectories in the $(\theta,P)$-plane follow horizontal lines, because $P$ is constant. Solutions can behave in two different ways depending on what power they have. This divides the phase plane into two separate regions; trajectories with $P > 4$ and $0 < P < 4$. Solutions in these two regions are explained, respectively, by the same argument used for the two types of solution before. A quick overview is that trajectories with $P > 4$ never touch $P_{\text{min}}$ whilst trajectories with $0 < P < 4$ do. This is shown in figure 10.

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Figure 10: *This figure gives examples of trajectories with $\gamma$ equals zero and $c$ greater than one. It shows two different trajectories, one above the separatrix and one below. The separatrix is given by the (blue) dotted line.*

5.3.4 Existence of a separatrix

A separatrix corresponds to a path with $P = 4$ and has been included by the dotted (blue) line in figure 10.

5.3.5 Trajectories in the ($\theta, \Delta$)-plane

Having found the solutions in the ($\theta, P$)-plane we can compare trajectories in the ($\theta, \Delta$)-plane. An expression for the trajectories is given by rearranging equation 5.2 to get;

$$\Delta^2 = P^2 - 4\cos^2 \theta. \tag{5.7}$$

Figure 11 shows how trajectories in the ($\theta, P$)-plane relate to those in the ($\theta, \Delta$)-plane. Firstly we see that trajectories with $P > 2(c + 1)$ oscillate their value of $\Delta$, appearing to be pulled towards and then away from the equilibrium point at
\( \theta = \Delta = 0 \). Trajectories with \( 0 < P < 4 \) oscillate around the trivial equilibrium point at \( \theta = -\pi/2 \) and \( \Delta = 0 \) in what can be described as a clockwise direction. From the figure it can be seen how similar trajectories with \( c = 1 \) are to those with \( c > 1 \).

![Figure 11: This figure shows trajectories with \( \gamma \) equals zero and \( c \) equal to one in \((\theta, \Delta)\)-space. These are examples of trajectories corresponding to figure 8.](image)

### 5.4 \( c \in (0, 1) \)

Trajectories with \( c \) between zero and one have been kept until last. This is because the analysis is made a little more confusing by the fact that a trajectory can be in either the upper or lower regions. These were defined in section 3 but we will remind ourselves before we continue. A trajectory is said to be in the upper region if the following equation is true;

\[
\sqrt{P^2 - \Delta^2} = 2 \cos \theta + 2\sqrt{c^2 - \sin^2 \theta}.
\]  

(5.8)
A trajectory is said to be in the lower region if;

\[
\sqrt{P^2 - \Delta^2} = 2 \cos \theta - 2 \sqrt{c^2 - \sin^2 \theta}.
\] (5.9)

In both regions \( \theta \in (-\arcsin c, \arcsin c) \). Despite the analysis being slightly more complicated the results in this section are arguably more interesting. We will find that what is known as a self-trapped state corresponds to one class of solutions.

5.4.1 **Trajectories switch between the upper and lower regions at the boundaries?**

We begin by discussing something unique to trajectories with \( c \in (0, 1) \). Section 3 explained how trajectories could only change what region they are in by touching the boundaries of \( \theta \). Despite being able to work out which region a trajectory switches to and from at certain boundaries that is not necessary information here. All we need agree on is that the switch between regions will happen if a trajectory reaches \( \theta = -\arcsin c \) or \( \theta = \arcsin \).

5.4.2 **The power is bounded below**

In both the upper and the lower region we can find lower bound of \( P \), \( P_{\min} \). In the upper region;

\[
P_{\min} = 2 \cos \theta + 2 \sqrt{c^2 - \sin^2 \theta}.
\] (5.10)

In the lower region;

\[
P_{\min} = 2 \cos \theta - 2 \sqrt{c^2 - \sin^2 \theta}.
\] (5.11)

Both of these are given by the upper and lower part of the ‘egg-shaped’ (orange) curve in the figure 12. An important thing to keep in mind is that equation 5.10 gives the lower boundary of the power for trajectories in the upper region, NOT
the lower region. Therefore, a trajectory can pass through this curve if it happens to be in the lower region, i.e. it obeys equation 5.9. Remembering this will help stop confusion when observing a solution passing below an 'apparent' lower boundary. To make this more clear, dashed lines are used wherever possible when a trajectory is in the lower region in figure 12.

5.4.3 Direction of trajectories

Just as in the previous sections, direction of trajectories depends on what the sign of $\Delta$ is. When in the upper region the trajectory follows the same rules as for $c > 1$ and $c = 1$. In other words, when $\Delta > 0$ the variable $\theta$ is increasing and when $\Delta < 0; \theta$ is decreasing. However, in the lower region these directions are reversed. This makes sense if we consider a trajectory in the upper region travelling with increasing $\theta$. When it reaches the boundary $\theta = \arcsin c$ the trajectory switches to the lower region. It cannot continue to move in the same way or it would pass through the boundary. Therefore it must switch direction, assuming it has not reached an equilibrium point.

5.4.4 Behaviour of trajectories in the $(\theta,P)$-plane

Trajectories in the $(\theta,P)$-plane will still follow horizontal lines, because $P$ is constant. The same arguments can be applied as for the previous sections, explaining both regions separately. However, this time three different types of solution are found. These can be categorised by; $2(1 - c) < P < 2\sqrt{1 - c^2}$, $2\sqrt{1 - c^2} < P < 2(1 + c)$ and $P > 2(1 + c)$. Note that the lowest possible power of a trajectory is given by the minimum of equation 5.11 to be $P = 2(1 - c)$. This corresponds to one of the two equilibrium points at $\theta = 0$. Also $P = 2\sqrt{1 - c^2}$ is the value of both $P_{\min}$ at the boundaries of $\theta$. Examples of such trajectories are labelled 'Stable 1' in figure 12.
Trajectories with $2(1 - c) < P < 2\sqrt{1 - c^2}$ have a lower power than the minimum value of $P_{\text{min}}$ in the upper region. This means that they must remain in the lower region forever. For this to happen the can never reach the boundaries of $\theta$. Instead the trajectory oscillates back and forth between to points where $\Delta = 0$. These are labelled 'Stable 2' in figure 12.

Trajectories with $2\sqrt{1 - c^2} < P < 2(1 + c)$, given in figure 12 by 'Stable 3', do reach the boundaries of $\theta$ where they switch between regions. When in the lower region the trajectories never touch $P_{\text{min}}$ because their power is above that of the curve. Therefore the path will travel from one boundary to the other. Trajectories do however touch $P_{\text{min}}$ in the upper region. The path cannot cross this barrier and the trajectory stops for an instance before reversing direction back towards the boundary of $\theta$.

Trajectories with $P > 2(1 + c)$ never touch $P_{\text{min}}$ in either region. They oscillate from side to side between the boundaries $\theta = -\arcsin c$ and $\theta = \arcsin c$ switching between regions and reversing direction each time they do.

5.4.5 Existence of two separatrices

Because there are three regions; there is a separatrix at two different values of the power. The one of these with $P = 2(c + 1)$ passes through the equilibrium with the highest power. The other separatrix corresponds to $P = 2\sqrt{1 - c^2}$.

5.4.6 Trajectories in the $(\theta, \Delta)$-plane

Having found the solutions in the $(\theta, P)$-plane we can compare trajectories in the $(\theta, \Delta)$-plane. An expression for the trajectories in the upper region is given by rearranging equation 5.8 to get;

$$\Delta^2 = P^2 - 4 \left( \cos \theta + \sqrt{c^2 - \sin^2 \theta} \right)^2,$$

(5.12)
Figure 12: This figure gives examples of trajectories with $\gamma$ equals zero and $c$ greater than one. It shows two different trajectories, one above the separatrix and one below. The separatrix is given by the (blue) dotted line.

with a similar result for trajectories in the lower region;

$$\Delta^2 = P^2 - 4 \left( \cos \theta - \sqrt{c^2 - \sin^2 \theta} \right)^2.$$  \hfill (5.13)  

We can compare the trajectories in the $(\theta,P)$-plane to those in the $(\theta,\Delta)$-plane. In doing this we see that stable trajectories with $2(1 - c) < P < 2\sqrt{1 - c^2}$ rotate around the equilibrium point at $\theta = \Delta = 0$. They do so in an anti-clockwise direction and their shape is demonstrated by figure 13. This is the opposite direction to the rotations in the phase planes for $c = 1$ and $c > 1$. In figure 14; trajectories form 'I' shape trajectories as they switch between regions, hitting $\Delta = 0$ in the upper regions. Lastly trajectories with $P > 2(c + 1)$ oscillate around a point with $\theta = 0$ but $\Delta \neq 0$. These are called self trapped states as they oscillate on one side of the $\theta$-axis. These are described in [7]. This are sketched in figure 15.
Figure 13: This figure shows a trajectory in $(\theta, \Delta)$-space that corresponds to the line labelled 'Stable 1' in the figure 12.

Figure 14: This figure shows a trajectory in $(\theta, \Delta)$-space that corresponds to the line labelled 'Stable 2' in the figure 12.
6 Phase plane analysis for $0 < \gamma < 1$

In the previous section we saw how we could use knowledge of the constant of motion $c$ to analyse trajectories with $\gamma = 0$. In this section we will refine some of these methods to find solutions with $\gamma \in (0, 1)$. The main difference comes from $P$ no longer being constant. Instead the rate of change of $P$ is proportional to $\Delta$. This section follows a similar structure to the previous section. It is recommended that section 5 be understood first, as some of the material explained for $\gamma = 0$ will be used here without much explanation.

6.1 Trajectories with $c > 1$

Start with the case of $c$ greater than one. Here $\theta$ is unbounded and we can reference back to the equation 5.2. Just as before we can find the lower boundary
of $P$, i.e. the line where $\Delta = 0$;

$$P_{\text{min}} = 2\cos \theta + 2\sqrt{c^2 - \sin^2 \theta}. \quad (6.1)$$

From the equation 4.2 we can see that equilibrium points lie on this curve at $\theta = -\pi + \arcsin \gamma$ and $-\arcsin \gamma$.

### 6.1.1 Expression for the trajectory paths

Next we must find what shape trajectories follow. To do this we use equations 2.7a and 2.7c combined with the equation above to show that trajectories obey the equation;

$$P(\theta) = 2\gamma \left( \theta + \arcsin \left( \frac{\sin \theta}{c} \right) \right) + K, \quad (6.2)$$

where $K$ is a constant to be determined. For a full derivation of this result refer to the appendix C. Having found a lower bound for the power $P$ it follows that when a trajectory touches this curve it cannot pass through it. Therefore the trajectory must stop and then continue to travel in the opposite direction, unless it reaches an equilibrium point.

The direction of trajectories along these paths is given by the argument used for $\gamma = 0$. Basically $\theta$ increases if $\Delta > 0$ and decreases if $\Delta < 0$.

### 6.1.2 What happens at boundaries?

Because we have chosen to restrict the phase difference to $-\pi \leq \theta < \pi$ we must change the value of $K$ as the trajectory crosses this boundary. This is because $P$ must remain continuous. Let $K = K_1$ before the trajectory crosses the boundary and $K = K_2$ after. Then if the trajectory is crossing from $\theta = \pi$ to $\theta = -\pi$; $P(-\pi) = P(\pi)$ implies $-2\gamma \pi + K_2$ equals $2\gamma \pi + K_1$. Therefore the value of $K$ increases by an amount $4\pi \gamma$. A similar argument can be used to say that $K$ must decrease by $4\pi \gamma$ when $\theta$ crosses from $-\pi$ to $\pi$. 

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6.1.3 Comparing the gradients of $P$ and $P_{\text{min}}$

The gradient of the $P_{\text{min}}$ line can be found to be;
\[
\frac{dP_{\text{min}}}{d\theta} = -2 \sin \theta \left( 1 + \frac{\cos \theta}{\sqrt{c^2 - \sin^2 \theta}} \right).
\]
(6.3)

Whereas the gradient of the trajectories is given by;
\[
\frac{dP}{d\theta} = 2 \gamma \left( 1 + \frac{\cos \theta}{\sqrt{c^2 - \sin^2 \theta}} \right).
\]
(6.4)

Therefore we can compare the gradients of the two of these lines. The gradient of the trajectory is less than the gradient of $P_{\text{min}}$ if $\gamma < -\sin \theta$. This means that trajectories travelling upwards can only intersect the lower boundary of $P$ when $\theta \in (-\pi + \arcsin \gamma, -\arcsin \gamma)$.

6.1.4 A separatrix passes through an equilibrium point

The equilibrium point found at $\theta = -\arcsin \gamma$ and $P = 2\sqrt{c^2 - \gamma^2} + 2\sqrt{1 - \gamma^2}$ has a separatrix passing through it. This separates the stable and unstable manifolds. It is possible to work out the value of $K$ for this separatrix by rearranging equation 6.2 and substituting in the values for the equilibrium point;
\[
K = 2\sqrt{c^2 - \gamma^2} + 2\sqrt{1 - \gamma^2} + 2\gamma \left( \arcsin \gamma + \arcsin \left( \frac{\gamma}{c} \right) \right).
\]
(6.5)

If this separatrix happens to cross the boundaries of $\theta$ then the other part of it is given with a $K$ value $4\pi \gamma$ less.

6.1.5 Behaviour of trajectories in the $(\theta,P)$-plane

Trajectories in the $(\theta,P)$-plane can be divided into two types. Unstable and stable. Unstable trajectories lie above the separatrix and touch $\Delta = 0$ in only one place. Stable trajectories lie below the separatrix and oscillate back and forth between two end points where $\Delta = 0$. Examples of such trajectories are shown in figure 16.
Figure 16: This figure shows how different trajectories with \( c > 1 \) appear in the \((\theta, P)\) phase plane. The (orange) ‘bell-shaped’ curve gives the minimum value of the power \( P_{\text{min}} \), through which trajectories cannot pass. Dashed (green) lines indicate the positions of the two equilibrium points. A blue dotted line represents a separatrix that separates the region of stability from the region of instability. An example of a stable trajectory is given by the (purple) line below the separatrix. The (red) line above the separatrix shows a typical unstable trajectory.
6.1.6 Behaviour of trajectories in the \((\theta, \Delta)\)-plane

Trajectories can be observed in the \((\theta, \Delta)\)-plane where similarities between the \(\gamma = 0\) case can be seen. The stable trajectories, shown in figure 17, rotate in a clockwise direction around the equilibrium point at \(-\pi + \arcsin \gamma\). These are separated by a separatrix from the unstable solutions that appear to have emerged from the other stable trajectories in the \(\gamma = 0\) case.

![Figure 17: This figure shows two different types of trajectories with \(c > 1\) in the \((\theta, \Delta)\) phase plane. Stable (purple) trajectories oscillate around the centre at \(\theta = -\pi + \arcsin c\). The other equilibrium point at \(\theta = -\arcsin \gamma\) is a saddle node. The dotted (blue) line passes represents the separatrix which surrounds the stable manifold. Unstable (red) trajectories tend off to \(\Delta \rightarrow \infty\).](image)

6.2 Trajectories with \(c = 1\)

Having already found a way to analysed trajectories when \(c > 1\) we can use a similar method to study solutions with \(c\) equal to one. Remember that the value
of the phase difference is bounded between $-\pi/2$ and $\pi/2$. This consists of using the lower boundary of $P$, $P_{\min}$. The non-trivial equilibrium point lies on this curve at $\theta = -\arcsin \gamma$ and the trivial equilibrium point is represented at $-\pi/2$.

6.2.1 Expression for the trajectory paths

In a similar way to before we can find an expression for the trajectories;

$$P(\theta) = 4\gamma \theta + K, \quad (6.6)$$

where $K$ is a constant to be determined. It can be seen that in this case trajectories follow straight lines with gradient $4\gamma$.

6.2.2 What happens at boundaries?

Let $K = K_1$ before the trajectory crosses the boundary and $K = K_2$ after. Then if the trajectory is crossing from $\theta = \pi/2$ to $\theta = -\pi/2$; the continuity of the power gives $P(-\pi/2) = P(\pi/2)$. This implies $-2\gamma \pi + K_2$ equals $2\gamma \pi + K_1$ Therefore the value of $K$ increases by an amount $4\pi \gamma$. A similar argument can be used to say that $K$ must decrease by $4\pi \gamma$ when $\theta$ crosses from $-\pi$ to $\pi$.

6.2.3 Comparing the gradients of $P$ and $P_{\min}$

The gradient of the $P_{\min}$ line can be found to be;

$$\frac{dP_{\min}}{d\theta} = -4 \sin \theta. \quad (6.7)$$

Whereas the gradient of the trajectories is given by;

$$\frac{dP}{d\theta} = 4\gamma. \quad (6.8)$$

Therefore we can compare the gradients of the two of these lines. The gradient of the trajectory is less than the gradient of $P_{\min}$ if $\gamma < -\sin \theta$. This means that
trajectories travelling upwards can only intersect the lower boundary of $P$ when $\theta \in (-\pi/2, - \arcsin \gamma)$. Such a trajectory would then stop momentarily and then start to travel downwards. This is therefore a stable trajectory.

6.2.4 Behaviour of trajectories in the $(\theta,P)$-plane

Trajectories in the $(\theta,P)$-plane can be divided into two types. Unstable and stable. Unstable trajectories lie above the separatrix and touch $\Delta = 0$ in only one place. Stable trajectories lie below the separatrix and oscillate back and forth between two end points where $\Delta = 0$. Examples of such trajectories are shown in figure 18.

6.2.5 Behaviour of trajectories in the $(\theta,\Delta)$-plane

Trajectories can be observed in the $(\theta,\Delta)$-plane where similarities between the $\gamma = 0$ case can be seen. The stable trajectories, shown in figure 19, rotate in a clockwise direction around the equilibrium point at $-\pi/2$. These are separated by a separatrix from the unstable solutions that appear to have emerged from the other stable trajectories in the $\gamma = 0$ case.

6.3 Trajectories with $\gamma < c < 1$

For $\gamma$, we saw that when $c$ is between zero and one there was a wider range of solutions. This we mainly due to the fact that a trajectory could be in the upper or lower regions. When $0 < \gamma < 1$ there are five categories of solutions. To see this remember that we can define $P_{\min}$ in both regions and these boundaries apply to trajectories in the respective regions. Note; trajectories switch between the regions when they reach the boundaries of $\theta$. 

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Figure 18: This figure shows how different trajectories with \( c = 1 \) appear in the \((\theta, P)\) phase plane. The (orange) "bell-shaped" curve gives the minimum value of the power \( P_{\text{min}} \), through which trajectories cannot pass. Dashed (green) lines indicate the positions of the one non-trivial equilibrium point. A blue dotted line represents a separatrix that separates the region of stability from the region of instability. An example of a stable trajectory is given by the (purple) line below the separatrix. The (red) line above the separatrix shows a typical unstable trajectory.
Figure 19: This figure shows two different types of trajectories with $c = 1$ in the $(\theta, \Delta)$ phase plane. The dashed (black) lines give the boundaries of $\theta$ where $|u_1||u_2| = 0$. Stable (purple) trajectories oscillate around the centre at trivial equilibrium point, $\theta = -\pi/2$. The other equilibrium point at $\theta = -\arcsin \gamma$ is a saddle node. The dotted (blue) line passes represents the separatrix which surrounds the stable manifold. Unstable (red) trajectories tend off to $\Delta \to \infty$. 

\[ \Delta \quad \theta \]

\[ -\pi/2 \quad 0 \quad \pi/2 \]
6.3.1 Expression for the trajectory paths

Next we find the expression for the path of the trajectory in these two regions;

\[ P(\theta) = 2\gamma \left( \theta + \arcsin \left( \frac{\sin \theta}{c} \right) \right) + K_1, \quad (6.9a) \]

\[ P(\theta) = 2\gamma \left( \theta - \arcsin \left( \frac{\sin \theta}{c} \right) \right) + K_2. \quad (6.9b) \]

where \( K_1 \) and \( K_2 \) are constants to be determined.

6.3.2 What happens at boundaries?

Because we have chosen to restrict the phase difference to \(-\pi \leq \theta < \pi\) we must change the value of \( K \) as the trajectory crosses this boundary. This is because \( P \) must remain continuous. Let \( K = K_1 \) in the upper region and \( K = K_2 \) in the lower region. Then continuity of the power at the boundary \( \theta = -\arcsin c \) implies \( K_1 - K_2 \) equals \( 2\pi \gamma \). A similar argument can be used to say that \( K_1 - K_2 \) equals \(-2\pi \gamma \) when \( \theta \) is at \( \arcsin c \).

6.3.3 Comparing the gradients of \( P \) and \( P_{\text{min}} \)

The gradient of the \( P_{\text{min}} \) line can be found to be;

\[ \frac{dP_{\text{min}}}{d\theta} = -2 \sin \theta \left( 1 \pm \frac{\cos \theta}{\sqrt{c^2 - \sin^2 \theta}} \right), \quad (6.10) \]

In the upper and lower regions respectively. Whereas the gradient of the trajectories is respectively given by;

\[ \frac{dP}{d\theta} = 2\gamma \left( 1 \pm \frac{\cos \theta}{\sqrt{c^2 - \sin^2 \theta}} \right). \quad (6.11) \]

Therefore we can compare the gradients of the two of these lines in the upper and lower regions separately. In both regions; the gradient of the trajectory is less than the gradient of \( P_{\text{min}} \) if \( \gamma < -\sin \theta \). This means that trajectories travelling
upwards can only intersect the lower boundary of $P$ when $\theta \in (-\pi/2, -\arcsin \gamma)$. Such a trajectory would then stop momentarily and then start to travel downwards. If this were to happen in the lower region then the trajectory would never enter the upper region. But if this should happen in the upper region the trajectory will touch the boundary at $\theta = -\arcsin c$ and enter the lower region.

6.3.4 Behaviour of trajectories in the $(\theta,P)$-plane

Trajectories in the $(\theta,P)$-plane can be categorised into five types. Two unstable and three stable. However not all of these trajectories can exist at the same time. In fact only four trajectories coexist. The first mix of trajectories is one unstable and three stable, shown in figure 20. The other combination of trajectories is two unstable and two stable. This is given in figure 21. The separatrix determines which configuration of solutions the system takes. Figure 20 shows the separatrix touching both boundaries of $\theta$. However, if the separatrix hits $P_{\text{min}}$ before it reaches the boundary $\theta = \arcsin c$ the solutions will look like figure 21.

6.3.5 Behaviour of trajectories in the $(\theta,\Delta)$-plane

When observing solutions in the $(\theta,\Delta)$-plane we use figures 22 - 24 to give the three types of stable trajectories separately. Trajectories labelled 'Stable 1' in figures 20 and 21 oscillates around the equilibrium point in the lower region when plotted in the $(\theta,\Delta)$-plane. An example of this type of trajectory is given by figure 22.

Figure 23 shows the trajectory labelled 'Stable 2'. It demonstrates the path touching one boundary. An ‘apple-shaped’ trajectory corresponding to 'Stable 3' is given in figure 24. This touches both boundaries of $\theta$.

The two possible unstable trajectories are sketched in figures 25 and 26.
Figure 20: This figure shows how different trajectories with $c \in (0,1)$ appear in the $(\theta, P)$ phase plane. Dashed (black) lines show the boundaries of $\theta$. The (orange) ‘egg-shaped’ curve gives the minimum value of the power $P_{\text{min}}$ in both the upper and lower regions. Trajectories in the upper region cannot pass through the top part of this line whilst trajectories in the lower region must remain above the bottom part. A dashed (green) line indicates the position of the two equilibrium points. A blue dotted line represents a separatrix that separates the region of stability from the region of instability. Examples of the three possible types of stable trajectory are given by the (purple) lines below the separatrix. The (red) line above the separatrix shows an unstable trajectory.
Figure 21: *This figure shows how different trajectories with $c \in (0, 1)$ appear in the $(\theta, P)$ phase plane when the separatrix, (blue) dotted line, touches $P_{\text{min}}$ in the lower region. In this case there are two types of unstable trajectories shown by (red) lines above the separatrix. There are also only two types of stable trajectories given by the (purple) lines below the separatrix.*

Figure 22: *This shows the trajectory labelled 'Stable 1' in figure 20.*
Figure 23: This shows the trajectory labelled ‘Stable 2’ in figure 20.

Figure 24: This shows the trajectory labelled ‘Stable 3’ in figure 20.
Figure 25: This shows the trajectory labelled 'Unstable 1' in figure 21.

Figure 26: This shows the trajectory labelled 'Unstable 2' in figure 21.
7 The Second Constant of Motion

7.1 Circles and Triangles

In this section we discuss an alternative approach for finding solutions to the governing equations. This method exploits the geometry of circles and triangles to find a useful parametrisation that in turn manages to reduce the dimension of the problem from two to one. To see this begin by recalling the equation defining $c$:

$$|u_1||u_2|(|u_1||u_2| - 2 \cos \theta) = c^2 - 1.$$ (7.1)

At this point it becomes clear why the choice of $c$ is useful in the format chosen. The link to circles comes by spotting that the above equation is in fact the cosine rule for a triangle with sides $1, |u_1||u_2|$ and $c$. Plus the interior angle $\theta$ opposite to the side of length $c$. Such a triangle is given in figure 29.

We have already shown that $c$ is a constant of motion. Also, of course, $1$ is also a constant. Therefore the result of this is that trajectories lie on circles with radius $c$ and a distance $1$ from the origin.

Now it seems clear why different values of $c$ may have different qualitative effects on the solutions. If $c < 1$ the circles do not contain the origin and therefore one value of $\theta$ may correspond to two different values for $|u_1||u_2|$. This also explains the boundaries on $\theta$. This is described by figure 28. When $c = 1$ the circle passes through the origin where $|u_1||u_2| = 0$. However, for $c > 1$ the circle contains the origin and every value of $\theta$ is possible. This is described by figure 27.

7.2 Parametrisation

Now that we have seen trajectories can be viewed to lie on circles, it might provide useful to define some parameter that tells us where on the circle a trajectory is at
Figure 27: In this diagram we see the relationship between $\theta$ and $|u_1||u_2|$ for $c > 1$. Notice how $\theta$ is not contained between any boundaries.

Figure 28: A diagram to show the relationship between $\psi$ and $|u_1||u_2|$ for $c \in (0, 1)$. In this case $\theta$ is clearly bounded. Also, one value of $\theta$ can correspond to two different values of $|u_1||u_2|$.
a given time. One thing to do is to use the arc angles of the circles corresponding to trajectories, so this is what we shall do. Define $\psi$ as the arc angle of the circle such that:

\[
c \sin \psi = |u_1||u_2| \sin \theta, \tag{7.2a}
\]
\[
c \cos \psi = |u_1||u_2| \cos \theta - 1. \tag{7.2b}
\]

Figure 29: This figure shows how $\psi$ can be defined in terms of $c$, $|u_1||u_2|$ and $\theta$.

This relationship is best understood through a diagram given in figure 29. For all times we choose to assume $\psi \in [0, 2\pi)$. We can rearrange these equations to get an expression for $|u_1||u_2|$;

\[
|u_1|^2|u_2|^2 = c^2 + 1 + 2c \cos \psi. \tag{7.3}
\]

This is shown in figure 30. An expression for $\theta$ can also be found;

\[
\tan \theta = \frac{c \sin \psi}{c \cos \psi + 1}. \tag{7.4}
\]

A sketch of $\theta$ against $\psi$ for different values of $c$ is shown 31.
Figure 30: In this diagram we see the relationship between $\psi$ and $|u_1||u_2|$ for different values of $c$.

Figure 31: This graph shows the relationship between $\psi$ and $\theta$ for different values of $c$. 
7.3 Equations for $\psi$

Now we have this new variable $\psi$ we want to find out what its rate of change with respect to $t$ is. To such an end we find;

$$\frac{d\psi}{dt} = \Delta. \quad (7.5)$$

This is a very elegant result that at first might seem very simple. After verifying this result it begins to become clear how this new variable might simplify the equations. By comparing with equation 2.7a we can write;

$$P = 2\gamma \psi + 2k, \quad (7.6)$$

where $k$ is, roughly speaking, a constant of motion. We use the word ‘roughly’ as care must be taken with defining $k$. This is due to the fact that we have chosen to constrain $\psi \in [0, 2\pi)$. This means that the value of $k$ will change as $\psi$ crosses between 0 and $2\pi$. We can substitute equations 7.5 and 7.6 into equation 2.7b to get;

$$\frac{d^2\psi}{dt^2} = 2\gamma (2\gamma \psi + 2k) + 4c \sin \psi. \quad (7.7)$$

Here we have used equation 7.2a with $\sqrt{P^2 - \Delta^2} = 2|u_1||u_2|$. Interestingly the equation 7.7 has a strong link to the differential equation of an oscillator. The first term on the right hand side corresponds to that of a forced simple harmonic motion. The second term corresponds to that on a non-linear oscillator, such as a pendulum. This equation is very hard, if not impossible, to solve explicitly. However, we can take advantage of a technique used for oscillator equations. By multiplying equation 7.7 through by $\frac{d\psi}{dt}$ and integrating with respect to $t$ we arrive at a first order differential equation;

$$\left(\frac{d\psi}{dt}\right)^2 = (2\gamma \psi + 2k)^2 - 8c \cos \psi + E \quad (7.8)$$
By using equation 7.4 it follows that:

\[ E = -4\left(c^2 + 1\right). \]  

(7.9)

Therefore it can be written:

\[ \frac{1}{4} \left(\frac{d\psi}{dt}\right)^2 = (\gamma\psi + k)^2 - 2c \cos \psi - c^2 - 1. \]  

(7.10)

This same result can be derived straight from equation 7.3. One benefit of this equation is that we can now plot the phase plane of \( \Delta = \frac{d\psi}{dt} \) against \( \psi \) for given \( c \). By plotting phase lines with different values of \( k \) shows that there is two different types of shapes to the graphs. Figure 32 shows that the phase plane can have a stable regions. However, this stable region doesn’t have to exist and rather the whole phase plane can be unstable, as shown in figure 33.

Figure 32: Matlab graph of the phase plane of \( \Delta = \dot{\psi} \) against \( \psi \). Here a stable region exists. The colour bar on the side indicates the \( k \) values of the trajectories.

We can also plot trajectories in the \((\psi, P)\)-phase plane in a similar way to how paths were found in the \((\theta, P)\)-phase plane. An expression for the minimum
Figure 33: Matlab graph of the phase plane of $\Delta$ against $\psi$. Here there is no stable region. The colour bar on the side indicates the $k$ values of the trajectories.

bound of $P$ can be found as a function of $\psi$.

$$P_{\text{min}}(\psi) = 2\sqrt{c^2 + 1 + 2c\cos\psi}.$$  \hfill (7.11)

From $P = 2\gamma\psi + 2k$ it is seen that there is a linear relationship between $P$ and $\psi$. Therefore trajectories follow straight lines. This is shown in figure 34.

### 7.4 Trajectories with the same $k$-value

For the case $\gamma = 0$ it is common place to plot solutions that have the same value of $P$. In doing this three specific values of $P$ become apparent. The first of this is, $P = 0$, the minimum value the power can take can take, a seemingly obvious result. Figure 35 shows the nature of trajectories when the power is low. The second value is $P = 2$. At this value the equilibrium point at $\theta = 0$ and $\Delta = 0$ bifurcates from a centre to a saddle node. Increasing $P$ past this point introduces self-trapping. This is given in figure 36. At this point in the phase plane; there
Figure 34: *This sketch shows an example phase plane in terms of the variables \( \psi \) and \( P \)*

are now a total of four equilibrium points. Three of these are centres and the other is a saddle node. The final value of \( P \) that stands out is \( P = 4 \). Beyond this point there exist trajectories that oscillate in one half of the phase plane, i.e. with either \( \Delta > 0 \) or \( \Delta < 0 \). However unlike the self-trapped states these do not oscillate round a center, and therefore do not form closed loops. Instead, \( \theta \) is not physically bounded and is either continuously increasing or decreasing depending on the sign of \( \Delta \). An example of this is given by figure 37. A similar analysis can be made for the case with \( \gamma > 0 \). However, instead of using \( P \), which is no longer constant for all trajectories, we use \( k \). To this end we find that there are three special values of \( k \). The analysis is best explained in the \((\psi, P)\) phase plane.
Figure 35: Matlab graph showing the \((\theta, \Delta)\) phase plane of trajectories with \(P \in (0, 2)\). The colour bar indicates the corresponding value of \(c\).

Figure 36: Matlab graph showing the \((\theta, \Delta)\) phase plane of trajectories with \(P \in (2, 4)\). The colour bar indicates the corresponding value of \(c\). This is the lowest values of \(P\) to give self-trapped states.
Figure 37: Matlab graph showing the (θ, Δ) phase plane of trajectories with $P > 4$. The colour bar indicates the corresponding value of $c$. In this region of $P$ there exist some trajectories that oscillate on one side of $Δ = 0$ but are unbounded in $θ$. 
The three values of $k$ are;

$$k = \sqrt{1 - \gamma^2} - \gamma (\pi + \arccos \gamma) , \quad (7.12a)$$

$$k = 2\sqrt{1 - \gamma^2} - 2\gamma (\pi - \arcsin \gamma) , \quad (7.12b)$$

$$k = -\gamma \pi . \quad (7.12c)$$

These correspond to;

- Bifurcation from centre to saddle node
- Separatrix at $c = 1$, no stable trajectories with $c < 1$
- Minimum value of $k$

Figure 38 shows two phase planes for trajectories with two different values of $k$. Unstable solutions can be seen instead of self-trap states.

8 Conclusions

In this paper we have seen that solutions of the equations for a PT-symmetric dimer with a constant gain-loss can be found in different ways. This involved finding a constant of motion in section 3 that we called $c$. Then equilibrium points were discussed in section 4 where bifurcation diagrams were plotted for the both $\gamma = 0$ and $\gamma \in (0, 1)$. These compared the positions of equilibrium points compared to the value of $c$. A threshold of PT-symmetry breaking was found at $\gamma = 1$ where there is a bifurcation of two equilibrium points to none. Such a threshold value is comparable with results found for experiments of a linear model [5].

When trying to find solutions the first method used was phase plane analysis. This was used to explain what trajectories did qualitatively. This took advantage of the reduce of the system from three to two dimensions given by $c$. Section 5
Figure 38: Matlab graph showing the $(\theta, \Delta)$ phase plane for two different values of the constant $k$. The figure (right) shows a value of $k \in (-\gamma \pi, \sqrt{1-\gamma^2} - \gamma (\pi + \arccos \gamma))$. The figure (left) shows the phase plane for $k = 2\sqrt{1-\gamma^2} - 2\gamma (\pi - \arcsin \gamma)$.
developed a method for solving the equations when $\gamma = 0$. This involved finding a lower bound for the power $P$ in the system in terms of the phase difference $\theta$. It was shown that we could explain all possible trajectories including a phenomena known as self-trapping. We saw how the constant $c$ greatly affected how the trajectories behaved on a qualitative level. The most interesting solutions were found for $c \in (0, 1)$, mainly due to the trajectory being permissible in to different regions which are described in section 3.

It was found that the method, first designed around $\gamma = 0$, in section 5, could be modified to explain solutions with $\gamma > 0$, in section 6. Finding an expression for the trajectories, i.e. finding $P$ as a function of $\theta$, lead to the explanation of all possible types of solutions. Unlike when $\gamma = 0$ it could be shown that unstable trajectories could be found for any value of $c$. The non-existence of the self-trapped state was also shown.

Finally the main result of the project was that the governing equations E.1a and E.1a were shown to be integrable. This was done in section 7 by making a link between the constant $c$ and the radius of circles. This meant that trajectories with the same value of $c$ where characterised by the same circles. By parametrising these circles, we defined a new variable $\psi$ that represented the arc angle of the circles. By writing equations in terms of this new variable; it was shown that a new constant of motion, which we named $k$, could be found. The system of equations could again be reduced, this time to one dimension. The whole system can be explained in one, pendulum-like, non-linear differential equation for $\psi$. Thus the system is integrable. The phase plane of this equation can show two different behaviours, namely one with a stable region and one without.
Having shown that the equations for the PT-symmetric dimer are indeed integrable, two separate directions of further work come to mind. One interesting extension would be to study the equations with $\gamma$ dependent on $t$. In doing this, several steps to finding solutions when $\gamma = \text{constant}$ will remain unchanged. One major difference would be that the equation 7.6 could not be deduced by integrating $dP/dt = 2\gamma d\psi/dt$. Instead an expression involving an integral will be found. This could then be substituted into the equations of motion to get a mixed differential and integral equation for $\psi$.

Another branch of future work might be to study the equations of a PT-symmetric dimer with a non-symmetric well depth. This essential changes the governing equations to:

\begin{align*}
    i\dot{u}_1 &= -u_2 - |u_1|^2 u_1 - i\gamma u_1 + E_1 u_1, \\
    i\dot{u}_2 &= -u_1 - |u_2|^2 u_2 + i\gamma u_2 + E_2 u_2.
\end{align*}

(9.1a)

(9.1b)

It would be interesting to see if one could extend the link with geometry to produce two constants of motion that make the equations integrable. Maybe the circles are instead some other shape such as an ellipse or maybe more likely a limacon.
References


A Calculations for section 2

A.1 Analysis for \( u_1 = 0 \)

In this appendix we analyse the system in the instance when \( u_1 = 0 \). It can be assumed that \( t = 0 \) at this point. Start by substituting \( u_1 = 0 \) into the governing equations to get;

\[
\dot{u}_1 = iu_2, \quad (A.1a)
\]
\[
\dot{u}_2 = (i|u_2|^2 + \gamma)u_2. \quad (A.1b)
\]

When plotting these complex variables in an Argand diagram; it can be seen that \( \dot{u}_1 \) is at a right angle to \( u_2 \). By Taylor expanding \( u_1 \) and \( u_2 \) about \( t = 0 \) we get;

\[
u_1(t) = itu_2(0) + o(t), \quad (A.2a)
\]
\[
u_2(t) = u_2(0) + t(i|u_2|^2 + \gamma)u_2(0) + o(t). \quad (A.2b)
\]

Therefore, when \( t \) is very close to zero the phase difference between the two wave guides is approximately \( \pm \pi/2 \). A similar result can be shown for \( u_2 = 0 \).

A.2 Finding the Polar form of the Governing Equations

By differentiating \( u_1 = |u_1|e^{i\phi_1} \) with respect to \( t \) we get;

\[
\dot{u}_1 = \left( \frac{d}{dt}|u_1| + i|u_1|\frac{d\phi_1}{dt} \right)e^{i\phi_1}, \quad (A.3)
\]

with a similar result for \( u_2 \). Substituting into equation 2.2a and dividing through by \( e^{i\phi_1} \) gives;

\[
\left( i\frac{d}{dt}|u_1| - |u_1|\frac{d\phi_1}{dt} \right) = -|u_2|e^{\phi_2-\phi_1} - |u_1|^2 - i\gamma|u_1|. \quad (A.4)
\]

We can separate the real and imaginary parts to this equation to give;

\[
|u_1|\frac{d\phi_1}{dt} = |u_2|\cos(\phi_2 - \phi_1) + |u_1|^3, \quad (A.5a)
\]
\[
\frac{d}{dt}|u_1| = -|u_2|\sin(\phi_2 - \phi_1) - \gamma|u_1|. \quad (A.5b)
\]
Similarly;

\[ |u_2| \frac{d\phi_2}{dt} = |u_1| \cos (\phi_2 - \phi_1) + |u_2|^3, \quad (A.6a) \]
\[ \frac{d}{dt} |u_1| = |u_1| \sin (\phi_2 - \phi_1) + \gamma |u_2|. \quad (A.6b) \]

On the right hand side of each of these equations, it can be seen that the system is not dependent on the two phase angles \( \phi_1 \) and \( \phi_2 \) independently. Rather, the system depends on the \( \theta = \phi_2 - \phi_1 \). I then follows that;

\[ |u_1||u_2| \frac{d\theta}{dt} = (|u_2|^2 - |u_1|^2) (|u_1||u_2| - \cos \theta). \quad (A.7) \]

### A.3 Governing Equations in terms of \( P \) and \( \Delta \)

This section shows how to get from the equations in polar form to describing the system in terms of \( P \) and \( \Delta \). This is done rather simply using the product rule for differentiation.

\[ \frac{d}{dt} |u_1|^2 = -2|u_1||u_2| \sin \theta - 2\gamma |u_1|^2; \quad (A.8a) \]
\[ \frac{d}{dt} |u_2|^2 = 2|u_1||u_2| \sin \theta + 2\gamma |u_2|^2. \quad (A.8b) \]

Adding and subtracting these equations results in;

\[ \frac{dP}{dt} = 2\gamma \Delta; \quad (A.9a) \]
\[ \frac{d\Delta}{dt} = 2\gamma P + 2\sqrt{P^2 - \Delta^2} \sin \theta. \quad (A.9b) \]

Here we have used that;

\[ \sqrt{P^2 - \Delta^2} = 2|u_1||u_2|. \quad (A.10) \]
B Calculations for section 3

B.1 Proving $c$ is constant

Using the product rule we can show that;

$$\frac{d}{dt} (|u_1||u_2|) = (|u_1|^2 - |u_2|^2) \sin \theta \quad (B.1)$$

Again, using the product rule combined with the chain rule we have;

$$\frac{d}{dt} (|u_1||u_2| \cos \theta) = \cos \theta \frac{d}{dt} (|u_1||u_2|) + |u_1||u_2| \frac{d}{dt} (\cos \theta)$$

$$= (|u_1|^2 - |u_2|^2) \sin \theta \cos \theta - \sin \theta |u_1||u_2| \frac{d\theta}{dt}$$

$$= |u_1||u_2| (|u_1|^2 - |u_2|^2) \sin \theta.$$

However, notice that also;

$$\frac{d}{dt} (|u_1|^2|u_2|^2) = 2|u_1||u_2| \frac{d}{dt} (|u_1||u_2|)$$

$$= 2|u_1||u_2| (|u_1|^2 - |u_2|^2) \sin \theta.$$

Thus, these two results only differ by a factor of 2. By integrating this result gives the following equation;

$$|u_1||u_2| (|u_1||u_2| - 2 \cos \theta) = \text{constant}. \quad (B.4)$$

For now we can call this constant $c^2 - 1$ with no loss of generality.

B.2 Proving that $c$ is real

In this part of the appendix we show that the conditions that $|u_1||u_2|$ is real and positive ensures $c$ is always real, and moreover can be called non-negative with no loss of generality. There are two possible solutions given by;

$$|u_1||u_2| = \cos \theta \pm \sqrt{c^2 - \sin^2 \theta}. \quad (B.5)$$
If \( c \) were to not be real then \( \sqrt{c^2 - \sin^2 \theta} \) would not be real. This would then imply that \( |u_1||u_2| \) is not real, a contradiction. Therefore \( c \) must be real. It also follows that there is no problem with requiring \( c \) to be non-negative as \( c \) only appears as \( c^2 \) in the equations.

### B.3 Trajectories with \( c = 0 \)

When \( c \) equals zero equation B.5 gives;

\[
|u_1||u_2| = \cos \theta \pm \sqrt{-\sin^2 \theta}. \tag{B.6}
\]

However, \( |u_1||u_2| \) must be real and the only way this is possible is if \( \sin \theta = 0 \), i.e. \( \theta = -\pi, 0 \). By then requiring that \( |u_1||u_2| \) be positive implies that \( \theta = 0 \). We can therefore say that any trajectory that has a \( c \)-value of zero has the following properties;

- \( \theta = 0 \)
- \( |u_1||u_2| = 1 \)

### B.4 Trajectories with \( 0 < c < 1 \)

For \( 0 < c < 1 \), the condition of \( |u_1||u_2| \) being real requires that;

\[
c^2 - \sin^2 \theta \geq 0. \tag{B.7}
\]

This inequality implies that either \( -\pi \leq \theta \leq -\pi + \arcsin c \), \( -\arcsin c \leq \theta \leq \arcsin c \) or \( \pi - \arcsin c \leq \theta < \pi \). However, by imposing that \( |u_1||u_2| \) be positive restricts trajectories with this value of \( c \) to the region \( -\arcsin c \leq \theta \leq \arcsin c \). For such trajectories it can be seen that;

\[
\cos \theta = \sqrt{1 - \sin^2 \theta} > \sqrt{c^2 - \sin^2 \theta}. \tag{B.8}
\]
This means that both the + and − solutions of equation B.5 are viable. We can therefore write that for trajectories with a $c$-value between zero and unity:

- $-\arcsin c \leq \theta \leq \arcsin c$
- $|u_1||u_2| = \cos \theta \pm \sqrt{c^2 - \sin^2 \theta}$

### B.5 Trajectories with $c = 1$

When a trajectory has $c$ equal to unity equation B.5 gives:

$$|u_1||u_2| = \cos \theta \pm \sqrt{1 - \sin^2 \theta}. \quad (B.9)$$

But $\sqrt{1 - \sin^2 \theta} = |\cos \theta|$. Thus, if we require $|u_1||u_2|$ to be positive then it must be the case that $-\pi/2 < \theta < \pi/2$. It also must be noted that any trajectory with $|u_1||u_2| = 0$ for any time $t$ must have a $c$-value equal to one. We can therefore say that for such a trajectory:

- $-\pi/2 < \theta < \pi/2$
- $|u_1||u_2| = 2 \cos \theta$, or
- $|u_1||u_2| = 0$

### B.6 Trajectories with $c > 1$

If $c$ is greater than one for a trajectory $\sqrt{c^2 - \sin^2 \theta}$ is real for all values of $\theta$. However to ensure that $|u_1||u_2|$ is positive; the − solution of equation B.5 cannot exist. Therefore, for a trajectory with $c$-value greater than unity:

- $-\pi \leq \theta < \pi$
- $|u_1||u_2| = \cos \theta + \sqrt{c^2 - \sin^2 \theta}$
C Calculations for section 6

In this appendix is the derivation of the expression given for the trajectories \( P \) in terms of \( \theta \). Start from the three equations:

\[
\frac{dP}{dt} = 2\gamma \Delta, \tag{C.1}
\]

\[
\frac{d\theta}{dt} = \Delta \left( 1 - \frac{2 \cos \theta}{\sqrt{P^2 - \Delta^2}} \right), \tag{C.2}
\]

\[
\sqrt{P^2 - \Delta^2} = 2 \cos \theta \pm 2\sqrt{c^2 - \sin^2 \theta}. \tag{C.3}
\]

Using the last two of these we can rewrite

\[
\frac{d\theta}{dt} = \Delta \left( \frac{\pm 2\sqrt{c^2 - \sin^2 \theta}}{2 \cos \theta \pm 2\sqrt{c^2 - \sin^2 \theta}} \right), \tag{C.4}
\]

\[
= \Delta \left( 1 \pm \frac{\cos \theta}{\sqrt{c^2 - \sin^2 \theta}} \right)^{-1}. \tag{C.5}
\]

But notice that;

\[
\frac{d}{d\theta} \left( \theta \pm \arcsin \left( \frac{\sin \theta}{c} \right) \right) = 1 \pm \frac{\cos \theta}{\sqrt{c^2 - \sin^2 \theta}}. \tag{C.6}
\]

Therefore, by the chain rule we can write;

\[
\frac{d}{dt} \left( \theta \pm \arcsin \left( \frac{\sin \theta}{c} \right) \right) = \Delta. \tag{C.7}
\]

Combining this with the equation for \( \frac{dP}{dt} \) gives;

\[
\frac{dP}{dt} = 2\gamma \frac{d}{dt} \left( \theta \pm \arcsin \left( \frac{\sin \theta}{c} \right) \right). \tag{C.8}
\]

Integrating with respect to \( t \) then gives;

\[
P = 2\gamma \left( \theta \pm \arcsin \left( \frac{\sin \theta}{c} \right) \right) + K, \tag{C.9}
\]

where \( K \) is the constant of integration.
D Calculations for section 7

Here we prove that $d\psi/dt = \Delta$. Begin with the equation;

$$c \cos \psi = |u_1||u_2| \cos \theta - 1.$$  \hfill (D.1)

By differentiating we see that;

$$-c \sin \psi \frac{d\psi}{dt} = \frac{d}{dt} (|u_1||u_2| \cos \theta).$$  \hfill (D.2)

We have already seen how to expand the right hand, using equation E.3d in the next section of the appendix for reference. This means that;

$$-c \sin \psi \frac{d\psi}{dt} = |u_1||u_2| (|u_1|^2 - |u_2|^2) \sin \theta.$$  \hfill (D.3)

Now notice that we can use, from the definition of $\psi; c \sin \psi = |u_1||u_2| \sin \theta$ to give the desired result.

E Equations

The appendix lists all equations from the rest of the paper and may be of use if the reader should wish to confirm results for themselves.

The equations for a PT-symmetric dimer are;

$$i\dot{u}_1 = -u_2 - |u_1|^2 u_1 - i\gamma u_1,$$  \hfill (E.1a)

$$i\dot{u}_2 = -u_1 - |u_2|^2 u_2 + i\gamma u_2.$$  \hfill (E.1b)

The equations for $|u_1|, |u_2|$ and $\theta = \phi_2 - \phi_1$ are;

$$\frac{d}{dt}|u_1| = -|u_2| \sin \theta - \gamma |u_1|,$$  \hfill (E.2a)

$$\frac{d}{dt}|u_2| = |u_1| \sin \theta + \gamma |u_2|,$$  \hfill (E.2b)

$$\frac{d\theta}{dt} = (|u_2|^2 - |u_1|^2) \left(1 - \frac{\cos \theta}{|u_1||u_2|}\right).$$  \hfill (E.2c)
Some useful expressions;

\[
\frac{d}{dt} (|u_1|^2) = -2|u_1||u_2| \sin \theta - 2\gamma|u_1|, \quad (E.3a)
\]

\[
\frac{d}{dt} (|u_2|^2) = 2|u_1||u_2| \sin \theta + 2\gamma|u_1|, \quad (E.3b)
\]

\[
\frac{d}{dt} (|u_1||u_2|) = (|u_1|^2 - |u_2|^2) \sin \theta, \quad (E.3c)
\]

\[
\frac{d}{dt} (|u_1||u_2| \cos \theta) = |u_1||u_2| (|u_1|^2 - |u_2|^2) \sin \theta. \quad (E.3d)
\]

The equations for \( P = |u_1|^2 + |u_2|^2 \) and \( \Delta = |u_2|^2 - |u_1|^2 \) are;

\[
\frac{dP}{dt} = 2\gamma \Delta, \quad (E.4a)
\]

\[
\frac{d\Delta}{dt} = 2\gamma P + 2\sqrt{P^2 - \Delta^2} \sin \theta, \quad (E.4b)
\]

\[
\frac{d\theta}{dt} = \Delta \left( 1 - \frac{2 \cos \theta}{\sqrt{P^2 - \Delta^2}} \right). \quad (E.4c)
\]

A relation between \( P \) and \( \Delta \) with \( |u_1||u_1| \) is;

\[
\sqrt{P^2 - \Delta^2} = 2|u_1||u_2|. \quad (E.5)
\]

The equations for \( c \) and \( \psi \) are;

\[
c \sin \psi = |u_1||u_2| \sin \theta, \quad (E.6a)
\]

\[
c \cos \psi = |u_1||u_2| \cos \theta - 1. \quad (E.6b)
\]

From this the expression for the constant \( c \) is;

\[
c^2 = 1 + |u_1||u_2| (|u_1||u_2| - 2 \cos \theta). \quad (E.7)
\]

An expression for \( |u_1||u_2| \) in terms of \( \theta \) is;

\[
|u_1||u_2| = 2 \cos \theta \pm \sqrt{c^2 - \sin^2 \theta}. \quad (E.8)
\]

Also, an expression for \( |u_1||u_2| \) is;

\[
|u_1||u_2|^2 = 1 + c^2 + 2c \cos \psi. \quad (E.9)
\]
The power of a trajectory in terms of $\theta$ can be written;

$$P(\theta) = 2\gamma \left( \theta \pm \arcsin \left( \frac{\sin \theta}{c} \right) \right). \quad (E.10)$$

The rate of change of $\psi$ is found to be;

$$\frac{d\psi}{dt} = \Delta. \quad (E.11)$$

$P$ and $\psi$ have the linear relationship;

$$P = 2\gamma \psi + 2k. \quad (E.12)$$

where $k$ is a constant of motion.

A second order differential equation for $\psi$ is;

$$\frac{d^2\psi}{dt^2} = 2\gamma (2\gamma \psi + 2k) + 4c \sin \psi. \quad (E.13)$$

A first order differential equation for $\psi$ is;

$$\frac{1}{4} \left( \frac{d\psi}{dt} \right)^2 = (\gamma \psi + k)^2 - 2c \cos \psi - 1 - c^2. \quad (E.14)$$