Equilibrium states and chaos in an oscillating double-well potential

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We investigate numerically parametrically driven coupled nonlinear Schrödinger equations modeling the dynamics of coupled wave fields in a periodically oscillating double-well potential. The equations describe, among other things, two coupled periodically curved optical waveguides with Kerr nonlinearity or Bose-Einstein condensates in a double-well potential that is shaken horizontally and periodically in time. In particular, we study the persistence of equilibrium states of the undriven system due to the presence of the parametric drive. Using numerical continuations of periodic orbits and calculating the corresponding Floquet multipliers, we find that the drive can (de)stabilize a continuation of an equilibrium state indicated by the change in the (in)stability of the orbit, showing that parametric drives can provide a powerful control to nonlinear (optical- or matter-wave-) field tunneling. We also discuss the appearance of chaotic regions reported in previous studies that is due to destabilization of a periodic orbit. Analytical approximations based on an averaging method are presented. Using perturbation theory, the influence of the drive on the symmetry-breaking bifurcation point is analyzed.

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I. INTRODUCTION

Parametric drives, i.e., external drives that depend on time and the system variables, have been used as a means to control and to maintain a system out of equilibrium [1,2]. The generation of standing waves when a liquid layer is subjected to vertical vibration, known as Faraday waves, is among the classical studies of parametrically driven instabilities [3]. The vertical vibration can sustain spatially localized temporally oscillating structures, commonly referred to as oscillons, such as in granular materials [4,5], Newtonian [6,7], and non-Newtonian fluids [8]. Ac parametric drives have also been predicted and have been shown to be able to sustain localized waves in a linear Schrödinger system, which, in the undriven case, would simply disperse [9]. This effect is referred to as dynamic localizations. Recently, it has been suggested theoretically [10,11] and has been shown experimentally [12,13] that periodically curved optical waveguide arrays can be an ideal system for realizations of dynamic localizations (see a recent review on theoretical and experimental advances in modulated photonic lattices that provide controls on fundamental characteristics of light propagations [14]). Such a localization has been used in quantum physics in the context of Bose-Einstein condensates to reduce or even to completely suppress quantum tunneling of particles trapped in a potential well by shaking the potential back and forth (see the review [15]). The tunneling suppression in a series of potential wells using the method has been shown experimentally [16–18]. These rather counterintuitive observations that shaken quantum systems expand less than more were explained theoretically in Refs. [9,19,20] where it was shown that the time-dependent coupling between lattice sites is described effectively by a (time-independent) zero-order Bessel function. The dynamic localization occurs at points of shaking parameters that correspond to vanishing the effective coupling.

Complementing the seminal finding of dynamic localization, coherent destruction of tunneling predicted in Refs. [21,22] has become an important phenomenon in the study of quantum dynamical control. Although dynamic localization occurs in infinite-dimensional systems, system boundaries play an important role in the coherent destruction of tunneling. The internal relationship between the two phenomena was discussed recently in Ref. [23] where it was shown that both phenomena can be interpreted as a result of destructive interference in repeated Landau-Zener level crossings. A recent experiment on strongly driven Bose-Einstein condensates in double-well potentials formed by optical lattices provided a direct observation of dispersion suppression in matter waves [24]. In the context of optical physics, the idea of tunneling destruction in couplers, i.e., dual-core waveguide arrays, was proposed in Ref. [25] and later was implemented experimentally in Refs. [26,27]. A sketch of the physical setups is shown in Fig. 1.

The phenomena of dynamic localization and coherent destruction of tunneling above originally were studied in the linear regime, and only rather recently, they were proposed as a powerful tool in nonlinear systems to manage, e.g., the dispersion of an atomic wave packet [28] and nonlinear gap solitons [29] and to control the dynamics of bright solitons [30–32]. In linear systems, the parameter values for tunneling suppression are isolated degeneracy points of the quasienergies. When nonlinearity is present in the systems, the tunneling can be suppressed in a relatively wide interval of parameter values [33]. It is also interesting to note that periodic driving may enhance tunneling as opposed to preventing it [34,35], which has recently been shown experimentally [36].

Despite the many important papers mentioned above, here, we study the effects of parametric drives in the systems sketched in Fig. 1 from a dynamical system point of view. In that regard, the present paper aims at studying the effects of nonlinearity on the aforementioned tunneling enhancements or
suppression. In particular, we consider the persistence, i.e., the existence and stability of equilibrium states when the potential is periodically oscillating. It will be shown that the parametric drive, in general, does not influence the existence of an equilibrium, which becomes a relative periodic orbit due to the drive but, indeed, may alter the stability. Destabilization of a periodic orbit will lead to the appearance of chaotic regions, i.e., due to separatrix chaos which is typical for chaos in Hamiltonian systems [37, 38]. The presence of chaos in the system discussed herein has been reported before [39] (see also Ref. [40] for a review). Here, we will consider the transition from the ordered to the chaotic states (or vice versa) due to the dynamics of an equilibrium under the effect of the parametric drives.

As discussed in the theoretical [41, 42] and experimental [43, 44] studies of Josephson tunneling of Bose-Einstein condensates in a double-well potential, the dynamics of the macroscopic quantum density and phase difference between two condensates is described by two coupled nonlinear ordinary differential equations. Depending on a control parameter, one can obtain qualitatively different dynamical behaviors that can be explained well by classical bifurcation theory. In general, there are three types of fixed points in the phase portraits. Using notations in the limit of no coupling between the condensates, the equilibria can be written symbolically as 

\[ u_1 = 0 \] and 

\[ u_2 = 0 \]

where \( u_1 \) and \( u_2 \) represent the densities of the condensates, and + and − represent 0 and π phases of the wave functions in the two potential minima, respectively, and 0 represents the case when the density vanishes in the well. The second equilibrium state is also commonly referred to as a self-trapped state because the density difference between the wells oscillates about a nonzero value. We will show that the self-trapped states, i.e., no-tunneling states, can be broken by the drive creating, possibly chaotic, tunneling between the two wells, i.e., a destruction of self-trapping with the relative phase of the fields oscillating coherently. In addition to that, by considering (without loss of generality) the self-focusing nonlinearity, i.e., attractive interaction between atoms in the case of Bose-Einstein condensates, we will also show that the symmetric [+] state, which is unstable beyond a critical norm, can be made stable by the drive, which we call a coherent reinstatement of tunneling. The antisymmetric [−] state, which is generally robust, can also be destabilized by the parametric drive. Note that our paper is also relevant to the study of Josephson tunneling between different internal states [45] or atom species [46] with an unfixed interaction between the states as the strength is controllable using a Feshbach resonance. Such setups exhibit interesting features that bear analogies with the present paper in the form of, e.g., transition from immiscibility to miscibility [47] and spatial demixing dynamics [48].

The paper is presented as follows. In Sec. II, we discuss the governing equation and the numerical methods used in the paper. In the following section, the destruction of self-trapped states due to a parametric drive is presented. We show that the destruction is caused by a period-doubling bifurcation. The instability of the self-trapped states is shown to create chaos. In Sec. IV, we study the persistence of symmetric states. We show that parametric drives are able to stabilize or to destabilize the states. A similar numerical discussion for the antisymmetric state is presented in Sec. V. The state which is always stable in the undriven case can become unstable due to parametric drives. Analytical calculations are presented in Sec. VI. We discuss the (de)stabilization intervals observed in the previous sections using an averaging method. The symmetry-breaking bifurcation point where the asymmetric and symmetric states merge is shown using perturbation theory to be affected by parametric drives. Conclusions are presented in the last section.

II. GOVERNING EQUATION AND NUMERICAL METHODS

Using a tight-binding approximation, the parametrically driven wave functions due to an oscillating potential, i.e., in periodically oscillating waveguides in nonlinear optics or horizontally shaken double-well magnetic traps in matter waves, are described by (see, e.g., Refs. [11, 13, 28])

\[ i\dot{u}_1 = \delta |u_1|^2 u_1 - qu_1 + ce^{-i\delta_0(z)} u_2, \]

\[ i\dot{u}_2 = \delta |u_2|^2 u_2 - qu_2 + ce^{i\delta_0(z)} u_1. \]

In the context of nonlinear optics, \( u_j \) is the optical field in the \( j \)th waveguide, \( c > 0 \) is the waveguide coupling coefficient (in units of 1/mm), \( q \) is the light propagation constant, and \( \delta > 0 \) is the nonlinearity coefficient [1/(W mm)]. The defocusing case \( \delta < 0 \) can be obtained immediately due to the staggering transformation \( u_j \to (-1)^j u_j \). The parametric drive is represented by the function \( \delta_0(z) = (n_2 \alpha / h)\tilde{\delta}_0(z) \), where \( \tilde{\delta}_0(z) \) describes the physical periodic curving profile, and \( h \) is the inverse of the light wave number. In this paper, we take \( \tilde{\delta}_0(z) = a\omega \sin(\omega z) \) with amplitude \( a \) in units of millimeters and waveguide curvature wave number \( \omega \). The constant time-independent quantity \( N = |u_1|^2 + |u_2|^2 \) associated with the field power is referred to as a norm herein. The parameters are taken in the vicinities of those used in Ref. [33]. The onset of, e.g., the existence and the stability of a solution certainly will depend on the parameter values, nevertheless, the results presented herein are qualitatively generic. Without loss of generality, one can scale \( \delta = \omega = 1 \). When \( a = 0 \), Eq. (1) is integrable (see, e.g., a review [49]).

013828-2
To study the persistence of an equilibrium in the presence of a parametric drive, we solve the governing equation (1) for periodic orbits. We, therefore, seek solutions satisfying $u_j(0) = u_j(T)$, where $T$ is the oscillation period. In the absence of a drive, equilibrium solutions of the equations clearly fulfill the relation. In the presence of drives, we look for the continuations of an equilibrium point by using shooting methods in real space or algebraic methods by discretizing the propagation direction variable. The latter method is more useful than the first when there is a bifurcation in the numerical continuation. In that case, we use a pseudoarclength continuation algorithm [50–52].

When a periodic orbit, say $U_0(z)$, is obtained, we also examine its stability by calculating its Floquet multipliers, which are eigenvalues of the monodromy matrix. This is obtained from solving a linearized equation about the solution $U_0(t)$:

$$i\dot{\xi}_1 = \delta (2|U_1|^2\xi_1 + U_1^2\xi_1^*) - q\xi_1 + ce^{-i\omega(t)}\xi_2,$$

$$i\dot{\xi}_2 = \delta (2|U_2|^2\xi_2 + U_2^2\xi_2^*) - q\xi_2 + ce^{i\omega(t)}\xi_1,$$

(2)

where $\xi_n(z)$ is a small perturbation to the periodic orbit $U_0(z)$. The linear system is integrated using a Runge-Kutta method of order four or a symplectic method. The monodromy matrix $M$ is then defined as

$$\begin{pmatrix}
\text{Re}[\xi_1(T)] \\
\text{Re}[\xi_2(T)] \\
\text{Im}[\xi_1(T)] \\
\text{Im}[\xi_2(T)]
\end{pmatrix} = M \begin{pmatrix}
\text{Re}[\xi_1(0)] \\
\text{Re}[\xi_2(0)] \\
\text{Im}[\xi_1(0)] \\
\text{Im}[\xi_2(0)]
\end{pmatrix}.$$

(3)

If the solution $U_0(z)$ is stable, all the multipliers must lie on the unit circle.

Writing $u_j = |u_j|e^{i\theta_j}$, it is convenient to represent a solution with its population imbalance $\Delta$ and phase difference $\theta$ between the light fields described as

$$\Delta = (|u_1|^2 - |u_2|^2)/N, \quad \theta = \phi_2 - \phi_1.$$  

(4)

One can show that $\Delta$ and $\theta$ satisfy parametrically driven sine-Gordon equations,

$$\Delta = 2c\sqrt{1 - \Delta^2} (\sin x_0 \cos \theta - \cos x_0 \sin \theta),$$

$$\dot{\theta} = \frac{2 \Delta c}{\sqrt{1 - \Delta^2}} (\cos x_0 \cos \theta + \sin x_0 \sin \theta) - \Delta N \delta.$$  

(5)

When there is no drive, i.e., $x_0 = 0$, we obtain the equations derived in Ref. [41]. In that case, (5) will have, at most, three fixed points given by $(\theta, \Delta) = (0,0), (\pm \pi,0)$, and $(0, \pm \sqrt{1 - (2c/\delta N)^2})$, which correspond to the symmetric, antisymmetric, and asymmetric states, respectively. In (1), the symmetric and antisymmetric states, respectively, are given by

$$u_1 = \frac{q + c}{\delta}, \quad u_2 = \frac{q - c}{\delta},$$

(6)

whereas, the asymmetric state is given by

$$u_1 = \sqrt{\frac{q + \sqrt{q^2 - 4c^2}}{2}}, \quad u_2 = \sqrt{\frac{q - \sqrt{q^2 - 4c^2}}{2}},$$

(7)

When $a = 0$, there are three topologically different phase portraits of (1) [41,42,53]. In Fig. 2(a), we show one of the types of the phase portrait in the $(\theta, \Delta)$ plane with $N = 2$ and $c = 10/11$. In this case, the symmetric equilibrium at $(0,0)$ is unstable (see the inset), whereas, there is a pair of stable equilibria lying on the vertical axis $\theta = 0$, which are the asymmetric ground states.

It is clear that the asymmetric state pair only exists when $c < \delta N/2$. The critical value $c = \delta N/2$ is a point of pitchfork (symmetry-breaking) bifurcation. At this point, the symmetric state changes stability, which is discussed in Sec. III below. The symmetric and antisymmetric states in the focusing case correspond to the antisymmetric and symmetric states, respectively, of the defocusing case due to the staggering transformation.

III. DESTABILIZATION OF ASYMMETRIC SELF-TRAPPED STATES

First, we consider the effect of the parametric drive on the asymmetric ground state.

When $a = 0$, there are three topologically different phase portraits of (1) [41,42,53]. In Fig. 2(a), we show one of the types of the phase portrait in the $(\theta, \Delta)$ plane with $N = 2$ and $c = 10/11$. In this case, the symmetric equilibrium at $(0,0)$ is unstable (see the inset), whereas, there is a pair of stable equilibria lying on the vertical axis $\theta = 0$, which are the asymmetric ground states.
Fixing the norm $N = 2$, the bifurcation diagram of the stationary $[+,+]$ and $[+,0]$ states is shown in Fig. 2(b) for varying coupling constant $c$. On the vertical axis is the propagation $q$, which numerically is a natural control parameter.

The symmetric $[+,+]$ state corresponds to branch $BD$. For a sufficiently large coupling, the state is known to be stable (see, e.g., Refs. [54,55] and references therein). When one decreases the parameter $c$, there will be a critical coupling below which the symmetric state is no longer stable. The equilibrium loses stability at a pitchfork (symmetry-breaking) bifurcation when $c = \delta N/2$ with an asymmetric state leading to “macroscopically quantum self-trapping” in the context of matter waves [41,42], i.e., point $A$ in the figure. The state corresponding to branch $AE$ can be viewed as the $[+,0]$ state. It is important to mention that the branch corresponds to two solutions having positive or negative values of $\Delta$. As shown in one of the insets in Fig. 2(b), the $BC$ branch corresponds to a periodic orbit encircling both the stable $[+,0]$ and the unstable $[+,+]$ states. The branch $FG$ corresponds to a periodic orbit encircling only one of the stable $[+,0]$ states. Both branches do not correspond to equilibria but are shown for the completeness of the analysis later. Finally, branch $HI$ corresponds to the stable antisymmetric $[+,−]$ states, that are given by [see (6)]

$$u_1 = -u_2 = \pm \sqrt{\frac{q + c - 1}{\delta}} e^{i\theta},$$

i.e., the branch is described by

$$q + c = \frac{N\delta}{2} + 1.$$  \hspace{1cm} (9)

Next, we consider the presence of a parametric drive in the system. Rather than showing the phase portraits in the $(\theta, \Delta)$ plane with a continuous time, it becomes more convenient to represent the solution trajectories in Poincaré maps (stroboscopic plots at every period $T = 2\pi$). In the recurrence map, a stable periodic orbit will correspond to an elliptical fixed point encircled by closed regions (islands). Setting $a = 0.05$, in Fig. 3, we show the recurrence maps of the system for norm $N = 2$ with two different values of $c$, i.e., $c = 0.974$ and 0.75, in the $(\theta, \Delta)$ plane.

The plots are obtained from various sets of initial conditions using direct numerical integrations of the governing equations (1). Note that, if Fig. 2(a) is an ordered dynamical picture, there are significant chaotic regions in Fig. 3. We know that the presence of chaos in the figures is at least caused by the $[+,+]$ state, which is a saddle point, i.e., separatrix chaos [38].

Looking at Fig. 3(a), at first glance, asymmetric trapped states in Fig. 2(b) are seemingly replaced by chaotic states. However, zooming in on the area around $\theta = 0$, $\Delta \approx 0.3$, it can be observed that there exist two small islands that are connected and can be obtained by one initial condition. We did not find the state’s counterpart with $\Delta < 0$. There is, instead, an island centered at $\theta = 0$, $\Delta \approx -0.8$ that does not correspond to trapped states. By checking the continuous evolution of an initial condition in this island, it is actually a quasiperiodic tunneling state [$\Delta(t)$ is not sign definite], that is locked to and oscillates coherently with the drive.

We observe that decreasing the coupling further will completely destroy asymmetric trapped states. Interestingly, by changing the coupling constant even further, one now has self-trapped states shown in Fig. 3(b) by a pair of small islands centered at $\theta = 0$, $\Delta \approx 0.6$, and $\Delta \approx -0.7$. With the decrease in the coupling, the island in Fig. 3(a) centered at $\theta = 0$, $\Delta \approx -0.8$ is transferred to that centered at $\theta = 0$, $\Delta \approx -0.98$ in Fig. 3(b). Moreover, two extra islands that are centered at $\theta = 0$, $\Delta \approx 0.8$ and $\Delta \approx -0.2$ can be observed.

Succeeding the numerical observations, the following questions arise: Why are there two islands for an asymmetric state with $\Delta$ positive in Fig. 3(a)? Why is there no self-trapped state with negative $\Delta$? How do new fixed points in Fig. 3(b) appear? We will show that the two topologically different maps in Fig. 3 are determined by the persistence of the equilibrium states.

We have solved the governing equations (1) for periodic orbits. The corresponding bifurcation diagram of Fig. 2(a) when $a \neq 0$ is shown in Fig. 4 where a rich bifurcation structure is evident.

Comparing Figs. 2(a) and 4, one can see that there is branch splitting. Branches $FG$ and $BC$ in Fig. 2(b) split into $F_jG_j$ and $B_jC_j$, $j = 1,2$ in Fig. 4. More importantly, there exists a small unstable section in branch $AF_1$, i.e., we
observe destabilization of asymmetric states. Note that there are two solutions along $AF_1$. The inset in the figure displays the Floquet multipliers of one of the solutions in the small section showing that it suffers from an instability due to a pair of multipliers leaving at $-1$. Floquet multipliers of the other solution present a similar behavior. The value of $c$ used in Fig. 3(a) belongs to the instability interval explaining why no single island corresponding to asymmetric trapped states is obtained in the Poincaré section.

The return of asymmetric trapped states in Fig. 3(b) when $c = 0.75$ is, indeed, in agreement with the bifurcation diagram in Fig. 4. In Fig. 3(b), the value of $c$ used belongs to the stable region $EF_2$. Note that there also are two solutions along the branch, which exactly correspond to the two stable fixed points at $\theta = 0$, $\Delta \approx 0.6$ and $\Delta \approx -0.7$ in the Poincaré map. For the value of $c$, there also are stable solutions in branch $F_1G_1$, which correspond to the islands centered at $\theta = 0$, $\Delta \approx 0.8$ and $\Delta \approx -0.2$.

The presence of only one fixed point at $\theta = 0$, $\Delta \approx -0.8$ in Fig. 3(a) without any counterpart in the upper half-plane can also be explained using the bifurcation diagram in Fig. 4. When $a = 0$, periodic orbits along branch $BC$ in Fig. 2(b) correspond to the fixed point. In the presence of the parametric drive, this branch splits into unstable $B_1C_1$ and stable $B_2C_2$ branches, and different from the case for branch $F_1G_1$, there is only one stable solution in branch $B_2C_2$. Thus, there will only be one family of stable coherent periodic orbits as seen in the Poincaré sections. The splitting itself is expected from the Floquet multipliers of the $BC$ branch as they are all at $+1$ (not shown here). This is also the case with branch $FG$. As for branch $AF$, no splitting is obtained as there is only one pair of Floquet multipliers at $+1$ (also not shown here).

From the inset of Fig. 4, the Floquet multipliers indicate that the solutions in the small unstable section in branch $AF_1$ suffer from a period-doubling bifurcation. To study the bifurcation further, we plot the homoclinic tangle of the solution with positive definite $\Delta(t)$ in branch $AF_1$ when $a = 0.04$ and $c = 0.974$ in Fig. 5(a).

It can be observed that, from the unstable solution with positive $\Delta$, the homoclinic tangle forms “loops” which exactly encircle the double islands in the inset of Fig. 3(a). Thus, it shows that there exists one stable period-doubling solution when $c = 0.974$ and $a = 0.04$ near the unstable solution with positive $\Delta$. This is confirmed by the numerical results shown in Fig. 5(b). Presented in the figure is the branch for period-doubling solutions ($JK$) near the symmetry-breaking bifurcation point $A$. It can be observed that the period-doubling solution is stable when it is close to the bifurcation point $J$. The value of $c$ used in Fig. 3(a) is in this stable region, explaining why two islands in the inset of Fig. 3(a) were obtained. With decreasing $c$, the solution also becomes unstable due to another period-doubling bifurcation.

In Fig. 6, we also have plotted the homoclinic tangle of the solution with negative definite $\Delta(t)$ in branch $AF_1$ when $a = 0.04$ and $c = 0.974$. Unlike the positive-definite state in Fig. 5(a), the invariant manifolds did not encircle any islands in Fig. 3(a). It may indicate that, for this state, the bifurcation is subcritical, which is proposed to be studied in the future.

The famous Smale-Birkhoff homoclinic theorem shows that having a homoclinic tangle guarantees that the system will have “horseshoe dynamics,” thus, potentially leading to
FIG. 6. (Color online) The same as Fig. 5(a) but for the unstable solution with negative $\Delta$.

chaos [56]. As the instability of a periodic orbit is related to the creation of homoclinic tangle as illustrated in Figs. 5(a) and 6, we, therefore, propose that Fig. 4 can be used to predict when we may obtain completely ordered trajectories. For $a = 0.04$, the figure suggests to us that there is no unstable branch when the coupling parameter $c$ is larger than $c \approx 1.17$. As illustrations, in Fig. 7, we plot the Poincaré maps of the governing equations when $a = 0.04$ for $c = 1.1$ and 1.2. When $c = 1.1$, from the inset of Fig. 7(a), we can still observe the presence of a chaotic region. However, for $c = 1.2$ where only stable solutions in branch $B_2C_2$ exist in Fig. 4, we observe that no chaotic region exists as shown in Fig. 7(b).

Finally, we would like to mention that the destabilization of the asymmetric states is rather generic, including the case of small $N$. One difference is that we only observed unstable solutions and did not see any splitting along the asymmetric state branch unlike that in Fig. 4. We have also studied the stability of asymmetric states for varying the driving amplitude $a$ and fixing the coupling constant $c$ and the power norm $N$. We show in Fig. 8 the stability curves of the states for two combinations of parameter values. Interestingly, we obtain intervals of driving amplitude in which the corresponding solutions are unstable due to a pair of multipliers leaving from $-1$, i.e., a period-doubling bifurcation.

IV. PERSISTENCE OF SYMMETRIC STATES

In Secs. II and III, we have seen that, in the undriven case, the symmetric states lose stability to asymmetric states when the coupling strength $c$ is below a critical value, see Fig. 2. It is natural to question whether it is possible to stabilize unstable symmetric states using the parametric drive. In Fig. 9, we show that it is, indeed, the case. Here (and in the following Sec. V) we only fix $c$ and $N$ and vary $a$. We did not compute any similar bifurcation diagram as those in Figs. 2 and 4 because the stability changes occur at relatively large values of $a$ and such a diagram will be rather completely different from that of the undriven case.

In Fig. 9(a), we depict the value of parameter $q$ as a function of the drive amplitude $a$ for two values of $N$. As a particular choice for clarity, we consider a symmetric state with a relatively large norm and small coupling such that, when there is no drive, the wave fields are weakly coupled and rather strongly unstable. It is interesting to note that, upon increasing parameter $a$, the continuation of the undriven symmetric state becomes stable. The stabilization is generic as we also observe the same phenomenon for other values of $N$ with the stabilization threshold almost independent of

FIG. 8. The stability curve of asymmetric states for varying driving amplitude $a$. Plotted is $q$ as a function of $a$ for $c = 0.2$ with $N = 1$ (upper) and $N = 0.7$ (lower). The dotted sections indicate unstable solutions.

013828-6
dynamics of the initial condition as solutions along the upper curve at \( a = N_c \) symmetric states (when undriven) with \( a = (\theta, \Delta) \). There is one pair of Floquet multipliers leaving and then unstable in a relatively wide interval of driving amplitude. The instability domain is finite, similar to the symmetric states reported in the previous section, in Fig. 11, we show that antisymmetric states can also become unstable when driven above a threshold value. The inset shows the trajectories of the dynamics in time in the \((\theta, \Delta)\) plane.

As for the dynamics of the instability, in Fig. 10(a), we depict a possible manifestation of the destabilization for \( u_1(0) = u_2(0) \) and \( N = 0.7 \). For this value of power, the state is stable when undriven as it is below the critical norm for a pitchfork bifurcation. When the state is driven, the relative phase is no longer localized, and the tunneling is completely suppressed, which can be called a nonlinear coherent destruction of tunneling similar to that discussed in Ref. [33].

For both the stabilization and the destabilization of symmetric states, when one increases the drive amplitude further, there will be other stability and instability windows, respectively.

V. DESTABILIZATION OF ANTISYMMETRIC STATES

Finally, we consider the antisymmetric \([+,-]\) state, which is stable in the undriven case. The state is generally robust. Nevertheless, in a similar fashion to the symmetric state in the previous section, in Fig. 11, we show that antisymmetric states can also become unstable when driven above a threshold value. The instability domain is finite, similar to the symmetric states
discussed in Sec. IV. When one increases the drive amplitude further, there will also be other instability windows.

In Fig. 11(b), we show the typical dynamics of an unstable $[+,−]$ state where we also obtain a nonlinear coherent destruction of tunneling similar to Fig. 10(b).

VI. ANALYTICAL RESULTS: STABILIZATION, DESTABILIZATION, AND SYMMETRY-BREAKING BIFURCATION POINTS

In this section, we analytically discuss the effects of parametric drives on the continuation of equilibrium states.

A method commonly used in almost all of the previous theoretical studies is to average the governing equation (1), which will yield (see, e.g., Refs. [9,11,19,20])

$$iμ_j = δ|μ_j|^2μ_j − qμ_j + cJ_0(a)μ_{2−j}, \quad j = 1, 2. \quad (10)$$

Here, $J_0$ is the Bessel function of the first kind. Hence, we obtain a coupler with an effective coupling constant $|cJ_0(a)|$. See also Refs. [57,58] for a different approach in effectively describing the transition dynamics using a transfer-matrix formalism.

As discussed in Secs. II and III above, there is a critical threshold at which the symmetric and asymmetric states change stability, i.e., the symmetry-breaking bifurcation point $c = N/2$. Note that $J_0(a)$ oscillates about 0 with the oscillation amplitude decreasing with $a$. Therefore, in the context of the average equation, the only possible way that the continuation of an equilibrium state changes stability is when the effective coupling crosses $±N/2$.

FIG. 11. (Color online) (a) The same as Fig. 8 but for the antisymmetric state with $c = 0.1$, $N = 2$ (upper), and $N = 1$ (lower). The inset shows the Floquet multipliers of a solution along the upper curve at $a = 2.5$. (b) The same as Fig. 10(b) for the antisymmetric mode with $N = 2$ and $a = 2.5$.

FIG. 12. (Color online) (a) The real and imaginary parts of the critical Floquet multipliers shown in black and red (light gray) solid curves, respectively, as a function of the solution norm $N$ with $a = 0.1$. The dashed curve is our approximation (19). (b) and (c) The typical time dynamics of the field intensity in each waveguide using the initial conditions from numerically exact periodic solutions with (b) $N = 2c = 0.4$ and (c) $N = 2c = 1.4$. The horizontal axis is normalized to the driving period $T = 2π$. 
For the asymmetric state discussed in Sec. III, one would expect that the state would persist and always be stable for any driving amplitude \( a \) because in the parameter region for the existence of the states, i.e., \( c < N/2 \), the effective coupling \( |cJ_0(a)| \) is always less than \( N/2 \). For the symmetric state studied in Sec. IV, the state would become stable when \( cJ_0(a) > N/2 \) or \( cJ_0(a) < 0 \). The latter is due to the staggering transformation as the state effectively becomes an antisymmetric one when \( cJ_0(a) \) changes sign. Finally, for the antisymmetric state analyzed in Sec. V, the state is expected to become unstable when \( 0 > cJ_0(a) > -N/2 \) also due to the staggering transformation. The analytical explanations of the stability switching above are rather in good agreement with the numerical results.

Despite the agreement, the averaged equation fails in describing several phenomena behind the stabilization and destabilization of the original governing equation, such as the period-doubling bifurcation that causes the destabilization of the asymmetric state instead of the state ceasing to exist and the branch splitting. Another failure of the averaged equation is in predicting the influence of parametric drives on the asymmetric trapped states in the undriven case, which implies that the pitchfork bifurcation should occur at \( c > N/2 \). It is because the effective coupling, i.e., \( |cJ_0(a)| \) at the point \( c = N/2 \) is less than half of the solution norm as \( J_0(a) < 1 \) for \( a > 0 \). Nevertheless, from Fig. 4, we obtained a contradiction that, at \( c = N/2 \), the symmetric state is still stable and the instability, i.e., the bifurcation of asymmetric states, occurs at a smaller coupling (or a larger norm when \( q \) is fixed). The average equation (10) is expected to be valid for \( \frac{q}{N} \sim N \), whereas, they are relatively of the same order in \( \frac{q}{N} \sim 1 \). To prove an alternative analytical discussion, we used a perturbation expansion.

We consider the equivalent equation (5). Studying the continuation of the symmetric state \((\Delta, \theta) = (0,0)\) in the presence of a parametric drive with small amplitude \( |a| \ll 1 \) at the bifurcation point of the undriven case \( c = N/2 \), we take the following expansion:

\[
\Delta = a\Delta^{(1)} + O(a^2), \quad \theta = a\theta^{(1)} + O(a^2),
\]

which, upon substitution into (5), yields

\[
\frac{d}{dz}\Delta^{(1)} = -N\theta^{(1)} + N\sin z, \quad \frac{d}{dz}\theta^{(1)} = 0.
\]

Looking for periodic solutions, we obtain

\[
\Delta^{(1)} = -N\cos z, \quad \theta^{(1)} = 0.
\]

Next, we study the stability of the periodic solution above. Let \((\Delta, \theta)\) be the periodic solutions (11) and (13). The linearization of (5) about it is then given by

\[
\begin{align*}
\dot{x} &= -\frac{2c\Delta}{\sqrt{1 - \Delta^2}}(\sin x_0 \cos \theta - \cos x_0 \sin \theta)x \\
-2c\sqrt{1 - \Delta^2}(\sin x_0 \sin \theta + \cos x_0 \cos \theta)y, \\
\dot{y} &= \frac{2c}{\sqrt{1 - \Delta^2}}\left(\frac{\Delta^2}{1 - \Delta^2} + 1\right) \\
&\times (\cos x_0 \cos \theta + \sin x_0 \sin \theta)x - N\delta x \\
&+ \frac{2\Delta c}{\sqrt{1 - \Delta^2}}(\pm \cos x_0 \sin \theta + \sin x_0 \cos \theta)y.
\end{align*}
\]

We will use the solution of the equation to construct a Floquet matrix that will determine whether or not the periodic solution is stable. It is then natural to expand the variables \( x(t) \) and \( y(t) \) in series as \( \Delta \) and \( \theta \) above, i.e.,

\[
x = x^{(0)} + a^2x^{(2)} + O(a^3), \quad y = y^{(0)} + a^2y^{(2)} + O(a^3).
\]

Substituting the expansions into the linearized equation (14), from terms of \( O(1) \), we obtain

\[
\begin{align*}
\dot{x}^{(0)} &= -Ny^{(0)}, \quad \dot{y}^{(0)} = 0.
\end{align*}
\]

From terms of \( O(a^2) \), we have

\[
\begin{align*}
\dot{x}^{(2)} &= -Ny^{(2)} + \frac{N^2}{2}\sin(2\Delta x^{(0)} \\
&+ \frac{N}{4}N^2\cos(2\Delta x^{(0)} + 1) + N\cos(2\Delta x^{(0)}), \\
\dot{y}^{(2)} &= \frac{N}{4}(3N^2\cos(2\Delta x^{(0)} + 1) - 1 + \cos(2\Delta x^{(0}})
\end{align*}
\]

Equations (16) are subject to the initial conditions that either \( x^{(0)}(0) = 1, y^{(0)}(0) = 0 \) or \( x^{(0)}(0) = 0, y^{(0)}(0) = 1 \), whereas, (17) is solved with the conditions \( x^{(2)}(0) = y^{(2)}(0) = 0 \).

Solving the linear equations for the two sets of initial conditions above and evaluating the values of the functions after one period \( T = 2\pi \), one will obtain the following Floquet matrix:

\[
M = \begin{pmatrix}
1 + \frac{a^2}{2}N^2\pi^2(1 - 3N^2) & -2N\pi + \frac{N^2\pi a^2}{6}[N^2(4\pi^2 + 3) + 3] \\
\frac{a^2}{2}N\pi(-1 + 3N^2) & 1 + \frac{a^2}{2}N^2\pi^2(1 - 3N^2)
\end{pmatrix} + O(a^3).
\]

From solving (16) and (17), one can note that the perturbation series (15) is nonuniform due to the presence of secular terms in the solution of (17). The approximation above breaks down at \( z \sim 1/a \), which limits the drive amplitude \( a \) for a given period \( T \).

Calculating the eigenvalues of \( M \), we obtain that

\[
\lambda_{1,2} = 1 \pm N\pi a\sqrt{1 - 3N^2 - O(a^2)}.
\]
VII. CONCLUSION

Through numerically solving parametrically driven coupled nonlinear Schrödinger equations describing the dynamics of wave fields in an oscillating double-well potential, we have shown that such parametric drives may stabilize or may destabilize the continuations of equilibrium time-independent states that are unstable or stable, respectively, in the undriven case. The analysis is performed by employing numerical continuations to find periodic orbits when varying a parameter and by calculating the corresponding Floquet multipliers of the states. Analytical calculations that accompany the numerical results using an averaging method and perturbation expansion have been presented where good quantitative agreement is obtained.

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