

Static semifluxons in a long Josephson junction with π -discontinuity points

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We investigate analytically a long Josephson junction with several π -discontinuity points characterized by a jump of π in the phase difference of the junction. The system is described by a perturbed-combined sine-Gordon equation. Via phase-portrait analysis, it is shown how the existence of static semifluxons localized around the discontinuity points is influenced by the applied bias current. In junctions with more than one corner, there is a minimum-facet-length for semifluxons to be spontaneously generated. A stability analysis is used to obtain the minimum-facet-length for multicorner junctions.

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I. INTRODUCTION

Superconductors are characterized by the phase coherence of the Cooper-pair condensate. Recent technological advances in the control of the phase near a Josephson junction have promoted research on the manipulation and phase biasing of such junction. Examples are the experimental preparation of superconductor-ferromagnet-superconductor (SFS) π junctions,¹ and superconductor-normal-metal-superconductor (SNS) junctions in which the charge-carrier population in the conduction channels is controlled.² These junctions are characterized by an intrinsic phase-shift of π in the current-phase relation or, in other words, an effective negative critical current.

An alternative branch of phase biasing is offered by the intrinsic anisotropy of unconventional superconductivity. A predominant $d_{x^2-y^2}$ pairing symmetry in high- T_c superconductors³ enables the possibility to bias parts of the circuit with a phase of π . Examples are the π -SQUID,^{4,5} tricrystal rings,³ the corner junction,⁶ and the zigzag junction.⁷ The latter two inspired the analytic investigation in the present work. These structures, of which neighboring facets in a Josephson junction can be considered to have opposite sign of the critical current, present intriguing phenomena such as the intrinsic frustration of the Josephson phase over the junction and the spontaneous generation of fractional magnetic flux near the corners. The presence of fractional flux, or semifluxons, has been considered before by several authors.⁸⁻¹⁵ In this work we present an analytic investigation of the existence and behavior of these semifluxons in an infinitely long Josephson junction with π discontinuities. We will introduce the model for these junctions and the method we use to analyze the semifluxons in a Josephson junction with π discontinuities in Sec. II. In Sec. III the results for one π discontinuity, the corner junction, is presented. Section IV discusses the case for two π discontinuities, in which we additionally introduce a minimum-facet-length between the discontinuities for which flux is spontaneously generated. In Sec. V, it is shown how the model is extrapolated to an increasing number of discontinuities in the infinitely long

Josephson junction. We use a stability analysis to discuss the existence of the semifluxons for this case. Section VI concludes the work.

II. MATHEMATICAL MODEL AND PHASE-PLANE ANALYSIS

To describe the dynamics of a long Josephson junction with π -discontinuity points a perturbed sine-Gordon equation is used:⁹

$$\phi_{xx} - \phi_{tt} = \theta(x) \sin \phi - \gamma + \alpha \phi_t, \quad (1)$$

where α is a dimensionless positive damping coefficient related to quasiparticle tunneling across the junction and $\gamma > 0$ is the applied bias current density, normalized to the junction critical current density J_c . The function θ takes the value ± 1 , representing the alternating sign of the critical current associated with the presence, or absence, of the additional π -phase shift.

Equation (1) is written after rescaling where the spatial variable x and time variable t are normalized by the Josephson penetration length λ_J and the inverse plasma frequency ω_p^{-1} , respectively. We consider static semifluxons, hence Eq. (1) reduces to

$$\phi_{xx} = -\frac{\partial U}{\partial \phi}, \quad (2)$$

where $U = -\theta(x)(1 - \cos \phi) + \gamma\phi$. For physically meaningful solutions ϕ and ϕ_x are required to be continuous at the discontinuity point.

The first integral of Eq. (2) is

$$\frac{1}{2} \phi_x^2 = -\theta(x) \cos \phi - \gamma\phi + C, \quad (3)$$

where C is the constant of integration. If θ would not depend on x , then the integral curves (“orbits”) of Eq. (3) form the *phase portrait* in the (ϕ, ϕ_x) phase plane.

Phase-plane analysis is particularly useful for the qualitative analysis of planar differential equations, see, for instance, Ref. 16 for an example involving a perturbed sine-Gordon equation. For a general introduction to phase-plane analysis see, for instance, Ref. 17.

In the present situation, however, θ does depend on x but in a special manner: θ takes only two values. Therefore there are two phase portraits that come into play, one with $\theta=1$ and another one with $\theta=-1$. Solutions of Eq. (2) are suitable combinations of orbits on these two phase planes. The position where the switch is made between the two phase planes is determined by the values of x where θ changes sign. With this convention in mind we will for simplicity speak about *the* phase portrait of Eq. (2). A similar approach is used by Walker¹⁵ to analyze a particular solution of Eq. (2) representing semifluxons in the case of $\gamma=0$. In his paper, Walker analyzes this situation using a combination of the potential functions U , i.e., $U=-(1-\cos\phi)$ and $U=(1-\cos\phi)$.

III. JUNCTIONS WITH A SINGLE π -DISCONTINUITY POINT

In a junction with a single π -discontinuity point, under certain conditions, a semifluxon is expected to be generated.³ With $\theta(x)$ given by

$$\theta(x) = \begin{cases} 1, & x < 0, \\ -1, & x > 0, \end{cases} \quad (4)$$

the parametrization of this semifluxon for $\gamma=0$ is^{9,12}

$$\phi(x) = \begin{cases} 4 \arctan \exp(x-x_0), & x < 0, \\ 4 \arctan \exp(x+x_0) - \pi, & x > 0, \end{cases} \quad (5)$$

where $x_0 = \ln(\sqrt{2} + 1)$. In the phase plane, the solution (5) is given by the combination of the curves with arrows in Fig. 1. For $x < 0$ we follow the solid curve starting at the origin up to d_1 , where $x=0$. From d_1 we switch flows and follow the dashed curve for $x > 0$ up to $(\phi/\pi, \phi_x) = (1, 0)$. This defines a semifluxon with a π -phase jump $\phi(\infty) - \phi(-\infty) = \pi$. The intersection of the trajectories for $x < 0$ and for $x > 0$ makes an angle, i.e., is transversal, which guarantees the persistence of the semifluxon when a bias current is applied. Later on it will be shown that this will remain the case up to $\gamma=2/\pi$.

In the phase plane, equilibria are the points that correspond to the maxima and minima of the potential U , i.e., $(\partial U/\partial \phi = 0, \phi_x = 0)$. When $\gamma=0$ two adjacent equilibria are connected by heteroclinic connections. Once we set $\gamma \neq 0$, the heteroclinic connections break and form homoclinic connections, i.e., connections between an equilibrium with itself. This opens the possibility for other solutions satisfying Eq. (2) than the semifluxon solution described above. As shown in Fig. 2, a semifluxon can be constructed by choosing d_2 or d_3 as the point where $x=0$. With these two discontinuity points, we obtain solutions with an overall phase jump of π , but containing humps as shown in Fig. 3. For $\gamma=0$ these constructions are not possible since the trajectories would pass through an equilibrium.

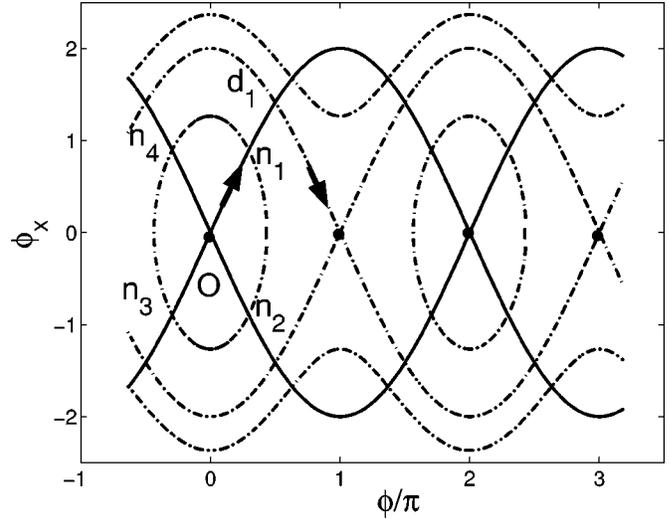


FIG. 1. The phase portrait of the system (2) for $\gamma=0$. The trajectories for $x < 0$ are indicated with bold lines, the trajectories for $x > 0$ with dashed lines. Any orbit of Eq. (2) switches at $x=0$ from bold to dashed. The semifluxon parametrized by Eq. (5) in the phase plane is indicated with arrows with d_1 is the corresponding position of $x=0$. O is the position of $(0,0)$. The definition of $d_1, n_1 - n_4$ are in the text.

The semifluxons with humps can be viewed as combinations of semifluxons and 2π fluxons.¹⁹ The semifluxon with d_2 as the position of $x=0$ consists of a semifluxon and a fluxon-antifluxon pair, while the semifluxon with d_3 for the corresponding position of $x=0$ consists of a semifluxon and a fluxon with opposite polarity. Because a fluxon and a semifluxon with opposite (like) polarity are attracting (repelling) each other, we can expect these semifluxons to be unstable. This means, that the solutions will not be static, but dissolve in time, resulting in the single semifluxon solution. For a further study of these semifluxons with humps, see Ref. 19.

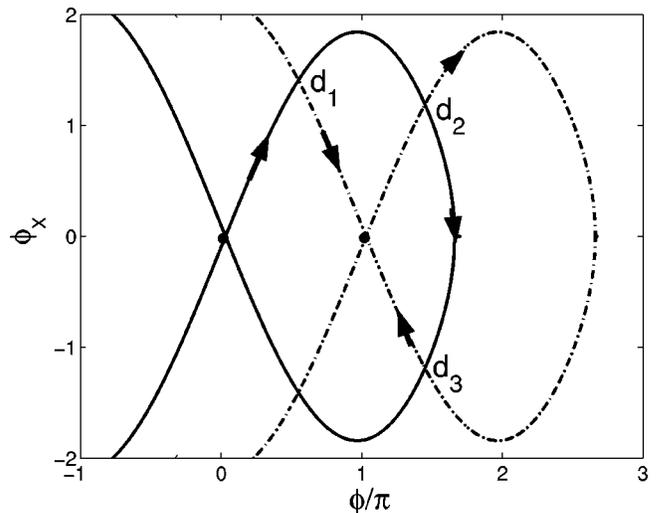


FIG. 2. The phase portrait of the system (2) for $\gamma=0.1$. For simplicity we only show the stable and unstable manifolds of the fixed points. Instead of d_1 , we might also take d_2 or d_3 for the position of $x=0$.

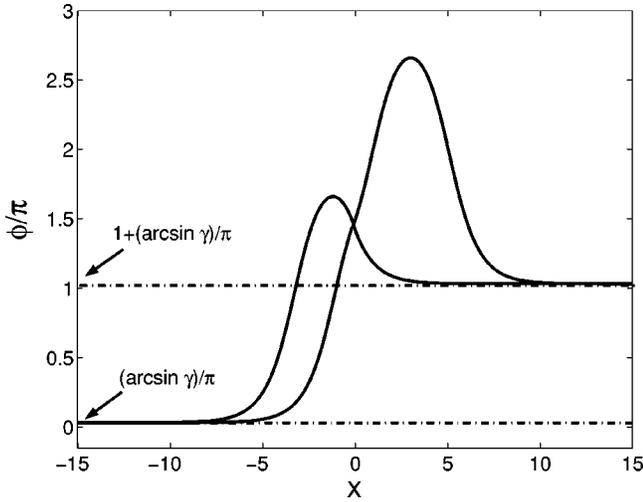


FIG. 3. Plot of the phase as a function of x (in units of λ_J) with d_2 and d_3 as discontinuity points. In this picture $\gamma=0.1$. The solution with higher hump is the phase with d_2 as the discontinuity point. Dash-dotted lines are the asymptotes of the solutions.

When increasing the normalized bias current γ , the homoclinic connections of the two equations will shrink and move apart. Therefore, for a certain value of γ , which we will denote as γ^* , d_2 and d_3 will coincide (see Fig. 4). When this happens, the solution with $x=0$ at d_2 corresponds to a semifluxon and a fluxon-antifluxon pair at infinity. For $\gamma > \gamma^*$, there is no solution with $x=0$ at d_2 that satisfies Eq. (2). Hence, only d_1 and d_3 can be used for the position of $x=0$, in that case.

To deduce the exact expression of γ^* , we consider the boundary conditions for the phase difference and magnetic flux at infinity

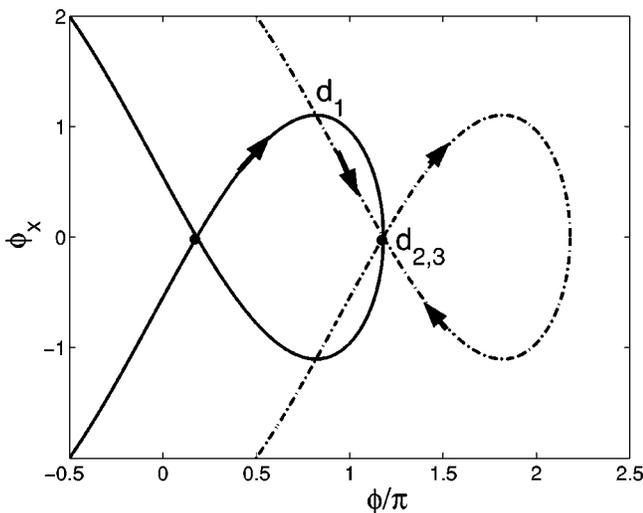


FIG. 4. Part of the phase portrait of Eq. (2) for $\gamma = \gamma^*$. At this value of γ d_2 and d_3 coincide.

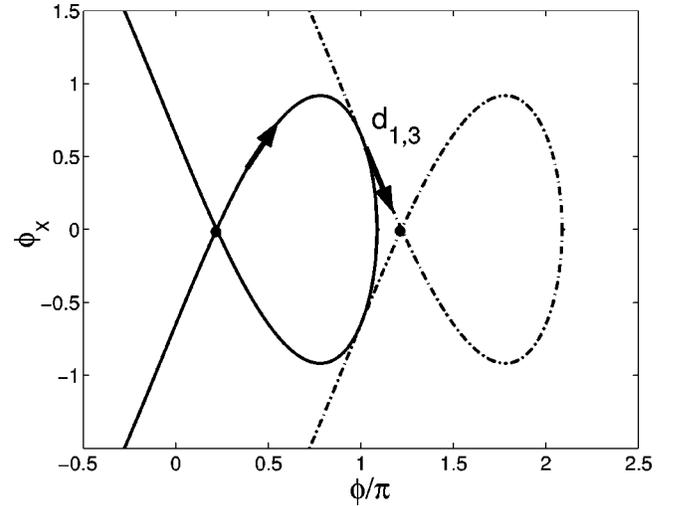


FIG. 5. The phase portrait of the system corresponding to the π kink solution for γ the critical current.

$$\lim_{x \rightarrow -\infty} \phi(x) = \phi_- = \arcsin(\gamma),$$

$$\lim_{x \rightarrow \infty} \phi(x) = \phi_+ = \pi + \arcsin(\gamma),$$

$$\lim_{x \rightarrow \pm\infty} \phi_x(x) = 0.$$

From the boundary conditions, the integral constant C of Eq. (3) is

$$C = \begin{cases} \cos \phi_- + \gamma \phi_-, & x < 0, \\ -\cos \phi_+ + \gamma \phi_+, & x > 0. \end{cases} \quad (6)$$

Imposing that $\phi_x(0^-) = \phi_x(0^+)$ and $\phi(0) = \pi + \arcsin \gamma$, we obtain the value of γ which gives the above condition

$$\gamma^* = \frac{2}{\sqrt{4 + \pi^2}} \approx 0.54. \quad (7)$$

When $\gamma > \gamma^*$, the point d_3 moves towards d_1 . At a certain value, the points d_1 and d_3 coincide. At that value of γ , the intersection of the trajectories of the system for $x < 0$ and $x > 0$ is tangential. The heteroclinic flow of the combined phase portrait is “smooth” (see Fig. 5). We then arrive at the edge of the static solution because as the trajectories intersect nontransversally, perturbations make them either nonintersecting or transversally intersecting.¹⁸ We will call the value of γ such that the intersection is nontransversal the critical current γ_c . Physically, if we apply a bias current larger than this critical current, the semifluxon attached at the π -discontinuity point will not be static.

A tangential intersection is achieved when at point $d_1 = d_3$,

$$\lim_{x \uparrow 0} \partial \phi_x / \partial \phi = \lim_{x \downarrow 0} \partial \phi_x / \partial \phi.$$

Noticing that $\partial \phi_x / \partial \phi$ is given by

$$\left. \frac{d\phi_x(x)}{dx} \frac{dx}{d\phi(x)} \right|_{x=0} = \left. \frac{\phi_{xx}}{\phi_x} \right|_{x=0} = \left. \frac{\pm \sin \phi - \gamma}{\phi_x} \right|_{x=0},$$

this condition is satisfied when $\sin \phi(0)=0$, i.e., $\phi(0) \in \{0, \pi\}$.

To obtain the value of ϕ_x at $x=0$, we use that

$$\int_{-\infty}^0 \phi_{xx} \phi_x dx = \int_{-\infty}^0 \phi_x \sin \phi - \gamma \phi_x dx, \quad (8)$$

which gives

$$\begin{aligned} \frac{1}{2} \phi_x(0^-)^2 &= -\cos \phi(0^-) + \sqrt{1-\gamma^2} \\ &\quad - \gamma[\phi(0^-) - \arcsin \gamma]. \end{aligned}$$

Here $\phi(0^-)$ is a shorthand for $\lim_{x \uparrow 0} \phi(x)$ and likewise for $\phi_x(0^-)$. Limits from the right are indicated by a “+.”

The same calculation for $x>0$ gives

$$\begin{aligned} -\frac{1}{2} \phi_x(0^+)^2 &= -\cos \phi(0^+) - \sqrt{1-\gamma^2} \\ &\quad - \gamma[-\phi(0^+) + \pi + \arcsin \gamma]. \end{aligned}$$

Because $\phi(0^-) = \phi(0^+) = \phi(0)$ and $\phi_x(0^-) = \phi_x(0^+)$, the two above expressions yield the critical current for the existence of static semifluxons

$$\gamma_c = \frac{2}{\pi} \approx 0.64. \quad (9)$$

No static solutions exist for γ above γ_c . This result is in agreement with the result of Kuklov *et al.*²⁰ that for $\gamma > \gamma_c$ the ϕ -fluxon changes the circulation back and forth while releasing 2ϕ fluxons. Using phase-plane analysis, we derive the maximum supercurrent from the existence of the static solution while Kuklov *et al.* derive it from the stability of the solution.

IV. JUNCTIONS WITH TWO π -DISCONTINUITY POINTS

The analytical discussion on junctions with two π -discontinuity points has been initiated by Kato and Imada.¹⁴ The junctions have a positive critical current for $|x| > a$ and a negative critical current for $|x| < a$. In this system there are two semifluxons with opposite polarity generated at the corners of the junction when a is relatively large. They conjecture that the magnetic flux is sensitive to the ratio $a = d/2\lambda_J$ where d is the distance of the two corners. We call the normalized distance of one corner to the next neighboring corner the normalized facet-length, which is $2a$ in our case. In this paper we consider the case when the semifluxons generated at the corners have opposite polarity.

Kato and Imada show numerically that the integrated magnetic flux, which is proportional to $\Delta\phi = |\phi(0) - \phi(\infty)|$, depends on a (see Fig. 6). When $a \gg 1$, they obtain $\Delta\phi = \pi$. The magnetic flux decreases when the facet length reduces. In the absence of a bias current, for $a \leq a_{\min}^{(2)} = \pi/4$, $\Delta\phi = 0$. Here $2a_{\min}^{(2)}$ is the minimum-facet-length necessary

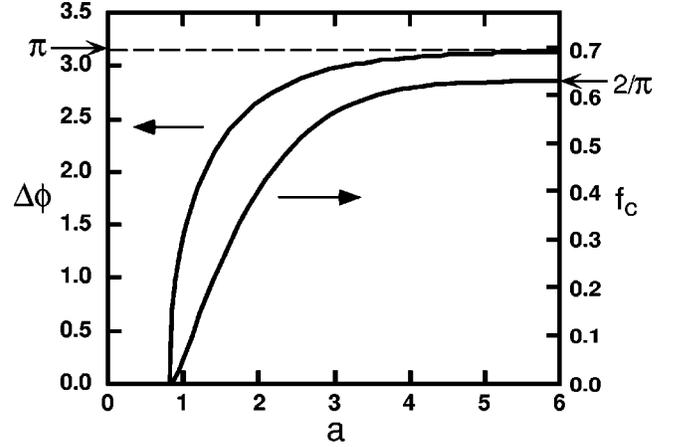


FIG. 6. $\Delta\phi$ is drawn as a function of a . The critical current f_c for vortices to change their orientation is also presented. The figure is taken with permission from Ref. 14.

to have a spontaneous flux generation when $\gamma=0$ (the superscript indicates the number of corners). The minimum-facet length for $\gamma \neq 0$ will be shown to be zero later on. In this section, we will show that the dependence of $\Delta\phi$ on a and $a_{\min}^{(2)}$ can be expressed explicitly when looking for the existence of the static semifluxons.

The phase portrait of the system without an applied bias current is given in Fig. 1. The semifluxon-antisemifluxon and the antisemifluxon-semifluxon states are represented by the trajectory $O-n_1-n_2-O$ and $O-n_3-n_4-O$, respectively. As $2a$ is the length of the middle junction, it is the pathlength of $n_1 \rightarrow n_2$ or $n_3 \rightarrow n_4$. If x would be replaced by t , as in the usual pendulum equation, it would be the time needed to go from $n_{1,3}$ to $n_{2,4}$.

Let M_p be the closed trajectory through points n_1 and n_2 . This flow represents the periodic motion of the pendulum equation. M_p crosses the ϕ -axes at the points $(\pm \Delta\phi, 0)$. Putting $\gamma=0$, from Eq. (3) M_p is implicitly given by the relation

$$\frac{1}{2} \phi_x^2 = \cos \phi - \cos \Delta\phi. \quad (10)$$

If $n_1 = (\phi^{it}, \phi_x^{it})$, then $\phi^{it} = \arccos(\cos \Delta\phi + \frac{1}{2} \phi_x^{it2})$.

The unstable and stable manifolds through n_1 and n_2 are given by

$$\frac{1}{2} \phi_x^2 = 1 - \cos \phi. \quad (11)$$

Hence, ϕ^{it} can be written as

$$\phi^{it} = \arccos[(\cos \Delta\phi + 1)/2]. \quad (12)$$

Now, it is straightforward to calculate the pathlength from n_1 to $(\Delta\phi, 0)$ which is exactly a . Using Eq. (10), we get

$$\begin{aligned}
 a &= \int_{\phi^{it}}^{\Delta\phi} \frac{d\theta}{\sqrt{2}(\cos\theta - \cos\Delta\phi)^{1/2}} \\
 &= \int_0^{\Delta\phi} \frac{d\theta}{\sqrt{2}(\cos\theta - \cos\Delta\phi)^{1/2}} \\
 &\quad - \int_0^{\phi^{it}} \frac{d\theta}{\sqrt{2}(\cos\theta - \cos\Delta\phi)^{1/2}}. \tag{13}
 \end{aligned}$$

Now consider

$$I = \int_0^{\phi^{it}} \frac{d\theta}{\sqrt{2}(\cos\theta - \cos\Delta\phi)^{1/2}}.$$

Using identity $\cos\theta = 1 - 2\sin^2(\theta/2)$, the integral becomes

$$I = \frac{1}{2} \int_0^{\phi^{it}} \frac{d\theta}{\sin(\Delta\phi/2) \sqrt{1 - \csc^2(\Delta\phi/2) \sin^2(\theta/2)}}.$$

If we let²¹ $\sin(\theta/2) = \sin(\Delta\phi/2) \sin\Phi$ such that the angle θ is transformed to $\Phi = \arcsin[\sin(\theta/2)/\sin(\Delta\phi/2)]$, the integral then becomes

$$I = \int_0^{\Phi^{it}} \frac{d\Phi}{(1 - k^2 \sin^2\Phi)^{1/2}} = F(\Phi^{it}, k)$$

with $k = \sin(\Delta\phi/2)$ and $\Phi^{it} = \arcsin[\sin(\phi^{it}/2)/\sin(\Delta\phi/2)]$. The function F is the incomplete elliptic integral of the first kind.²¹ Hence, we get that

$$a = F(\pi/2, k) - F(\Phi^{it}, k).$$

This is the explicit relation between $\Delta\phi$ and a when $\gamma = 0$. The plot is shown in Fig. 6. With this expression, we can see that

$$\lim_{\Delta\phi \rightarrow 0} a = \pi/4$$

because $k \rightarrow 0$ and $\Phi^{it} \rightarrow \pi/4$. This value is the minimum pathlength from $n_{1,3}$ to $n_{2,4}$ at the limiting point O , which is then the minimum-facet-length to have a semifluxon-antiseifluxon or an antiseifluxon-semifluxon at the corners.

When we start applying a bias current to the junction, a vortex and an antivortex are created at the corners even though the facet length is less than the minimum-facet-length for $\gamma = 0$. In other words, the minimum-facet-length of the junction when $\gamma \neq 0$ is 0. In the phase portrait, this can be seen from the fact that the equilibria of the system for $x < 0$ do not coincide with the ones for $x > 0$ when $\gamma \neq 0$, see Fig. 7. In the presence of the applied bias current, the magnetic flux $\Delta\phi$ is also influenced by γ . There are two different cases of the behavior of the semifluxons under the influence of a bias current. In the following, we will discuss the two cases separately.

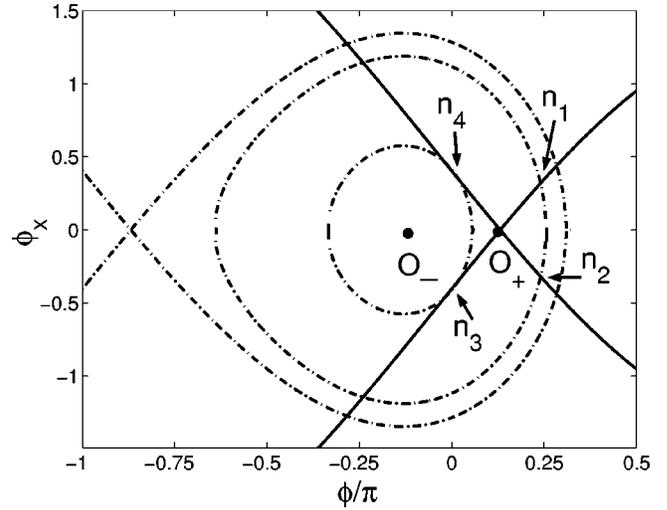


FIG. 7. The phase portrait of Eq. (2) when $\gamma = 0.4$.

A. Antiseifluxon-semifluxon case

When we start with a pair of antiseifluxon-semifluxon which we will here call a ϕ_2 solution, there is a critical value of the applied bias current to reorient the solution such that it becomes a semifluxon-antiseifluxon, here called a ϕ_1 solution. The flipping over from the ϕ_2 solution to the ϕ_1 solution has been discussed numerically in Ref. 14. For simplicity f_c is used to denote the critical bias current as in Ref. 14. In Fig. 6, f_c is drawn as a function of a . We will discuss the flipping-over process using phase-plane analysis.

The flipping process can be explained by considering the phase portrait of the system for $\gamma \neq 0$, for which a sketch is drawn in Fig. 7. When $\gamma \neq 0$, the origin O splits into O_- (an equilibrium of the system for $x < |a|$) and O_+ (an equilibrium of the system for $x > |a|$). The minimum-facet-length to have a ϕ_2 solution for $\gamma \neq 0$ now is given by the pathlength of $n_3 - n_4$. These points lie on the unstable and stable manifolds of the system for $x > |a|$. The flipping process happens because the minimum-facet-length to obtain a ϕ_2 -solution for a positive bias current increases when γ increases.

Therefore, given the facet length, increasing γ we arrive at a level of the bias current where the ϕ_2 solution ceases to exist, and the solution will switch to solution of type ϕ_1 . In Fig. 7 the ϕ_1 solution corresponds to the curve $O_+ - n_1 - n_2 - O_+$. Hence, $f_c(a)$ also shows the minimum-facet-length to have an antiseifluxon-semifluxon state for given positive γ . The exact expression of this relation can be given in terms of a modified integral elliptic function.

B. Semifluxon-antiseifluxon case

When we start with a semifluxon-antiseifluxon state, with a positive current as long as the bias current is less than $2/\pi$, the static semifluxons are attached at the corners for any facet length. If $\gamma > 2/\pi$ there is a limiting value of a , say $a_m(\gamma)$, such that for $a > a_m$ there is no static semifluxon-antiseifluxon state. a_m depends monotonically on γ which means that we can determine the critical current γ_c for given

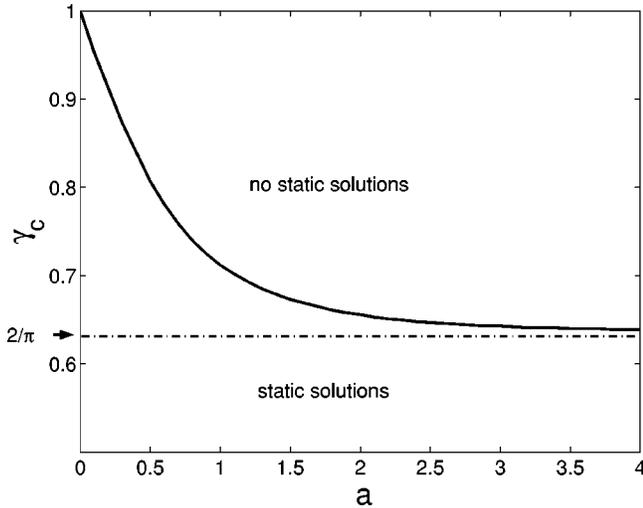


FIG. 8. Plot of the critical current γ_c as a function of half facet length a .

a . With a certain value of a and $\gamma > \gamma_c$, there is no static vortex-antivortex state. The plot of the relation between a and γ_c is presented in Fig. 8.

When the static solutions disappears, the solution becomes time dependent and starts flipping between the two types of semifluxons (the vortex-antivortex and the antivortex-vortex) while releasing 2π fluxons.

The difference between γ_c and f_c is that the applied bias current of γ_c is the minimum value of the current to pull the two semifluxons apart while f_c is the minimum current to collide the two semifluxons. This can be seen from the Lorentz force induced by the applied bias current. In the limit $a \rightarrow \infty$, both γ_c and f_c converge to $2/\pi$.

V. JUNCTIONS WITH MULTIPLE π -DISCONTINUITY POINTS

In Ref. 22, Goldobin *et al.* consider multicorner junctions. One of the problems they consider is to determine the minimum length such that semifluxon states do exist for $\gamma=0$. They have shown numerically that the minimum length varies as a function of the number of discontinuity points. In this section, we discuss the question of determining the minimum-facet-length $a_{\min}^{(N)}$ for a junction with $N\pi$ discontinuity points analytically. Recall that $a_{\min}^{(N)}$ denotes the minimum-facet-length in absence of a bias current. All the facet lengths are assumed to be equal.

One should be able to use existence analysis to determine the minimum-facet-length for multicorner junctions as we did for the case of two-corner ones, but it seems rather difficult. Therefore we use a stability analysis of the constant solution. This constant solution is the trivial state. The idea is mentioned in brief by Kuklov *et al.*²⁰ and Kato and Imada.¹⁴

Based on numerical simulations, we assume the following.

Once the trivial state is unstable, it creates semifluxons in an antiferromagnetic order. Experimentally, this antiferromagnetic ordering has indeed been found as the ground state of $\text{YBa}_2\text{Cu}_3\text{O}_{7-\text{N}}\text{-Nb}$ zigzag junctions.⁸

For $a < a_{\min}^{(N)}$, there is no spectrum with positive real part. When $a = a_{\min}^{(N)}$ there is a zero eigenvalue (discrete spectrum) which moves into the right half-plane when $a > a_{\min}^{(N)}$. It indicates that the trivial state becomes unstable and spontaneous fluxons are then generated.

The damping is assumed to be absent, i.e., $\alpha=0$, as it does not influence the value of the minimum-facet-length.

The mathematical details will be discussed elsewhere.

With the above assumptions the idea of getting the minimum-facet-length is by looking at an eigenvalue at zero. The starting equation is

$$\phi_{xx} - \phi_{tt} = \theta(x) \sin \phi, \quad (14)$$

which is Eq. (1) with $\gamma = \alpha = 0$. Here γ is taken to be zero because we calculate the minimum-facet-length in absence of a bias current.

Equation (14) admits $\phi^0 = k\pi$, $k \in \mathbb{Z}$ as the trivial solutions. We then linearize about ϕ^0 writing $\phi = \phi^0 + v(x, t)$, and retaining the terms linear in v :

$$v_{xx} - v_{tt} = \theta(x)v \cos k\pi. \quad (15)$$

We now make the spectral ansatz $v(x, t) = e^{\lambda t} u(x)$ which gives for u the equation

$$u_{xx} - \lambda^2 u = \theta(x)u \cos k\pi. \quad (16)$$

The real part of λ determines the stability of the trivial solution.

According to Ref. 23, the boundary of the essential spectrum is given by those eigenvalues λ for which there exists a solution to Eq. (16) of the form $u(x) = e^{i\zeta x}$, with ζ real. It follows that $\lambda = \pm \sqrt{\mp 1 - \zeta^2}$ and hence there are positive eigenvalues if $\lambda = \pm \sqrt{1 - \zeta^2}$. This explains that $k\pi$ is not a stable constant solution of the system if $\theta(\pm\infty) = \pm 1$.

With the result above, we conclude that there is no stable constant solution if $\theta(\infty) \neq \theta(-\infty)$. It means there is no minimum-facet-length of a long Josephson junction with an odd number of corners. For any facet length, we will always obtain semifluxons attached at the corners with a total phase jump $|\phi(\infty) - \phi(-\infty)| = \pi$.

Josephson junctions with an even number of π -discontinuity points could have a stable trivial solution. According to our second assumption, we need to compute the discrete spectrum. The stability of the trivial solution will depend on the facet length.

If we look at Eq. (16) (without losing generality we can take $k=0$), this equation belongs to the classical scattering problem.²⁴ This problem has been well discussed in quantum mechanics²⁵ where $\theta(x)$ is the potential function. According to Ref. 24, a discrete eigenvalue is a value of λ^2 for which the corresponding eigenfunction decays exponentially as $x \rightarrow \pm\infty$.

As an example, the case of four corner junctions is considered. The solution of Eq. (16) with the above requirement can easily be constructed by considering that we have five regions based on $\theta(x)$, i.e.,

$$u = \begin{cases} A_{1,2} \exp(\sqrt{1+\lambda^2}|x|), & |x| > 3a, \\ B_{1,2} \cos[\sqrt{1-\lambda^2}(|x|-2a)] + C_{1,2} \sin[\sqrt{1-\lambda^2}(|x|-2a)], & 3a > x > a, \\ D_1 \cosh(\sqrt{1+\lambda^2}x) + D_2 \sinh(\sqrt{1+\lambda^2}x), & |x| < a. \end{cases} \quad (17)$$

Next, we have to determine all the coefficients using the continuity conditions $u(c^-) = u(c^+)$ and $u_x(c^-) = u_x(c^+)$, $c = \pm a, \pm 3a$. Notice that the facet length is $2a$. In the matrix form, the eight linear homogeneous equations are written as

$$\Lambda(A_1 \ B_1 \ C_1 \ D_1 \ D_2 \ B_2 \ C_2 \ A_2)^T = 0, \quad (18)$$

with Λ is the coefficient matrix. To calculate the minimum-facet length, we take $\lambda = 0$.

The above system has nontrivial solutions, i.e., solutions that carry spontaneous flux, only if the determinant of the coefficient matrix vanishes. This leads to the equation

$$\begin{aligned} & \cosh^2(a) \{ 16[\cos^4(a) - \cos^2(a)] + 2 \} + 2 \cosh(a) \sinh(a) \\ & + 8[-\cos^4(a) + \cos^2(a)] - 1 = 0. \end{aligned} \quad (19)$$

As shown in Fig. 9 numerically, this equation has several solutions. The minimum-facet-length to obtain an antiferromagnetically ordered semifluxons is the smallest nonnegative root of Eq. (19). Then we conclude that $a_{\min}^{(4)} \approx 0.65$ (normalized to λ_J). The next roots correspond to stable orderings of the semifluxons and antisemifluxons other than the antiferromagnetic one. This is a subject for future work.

For a six corner junction, we find in this way that $a_{\min}^{(6)} \approx 0.56$. So far we have calculated some minimum-facet lengths which show a dependence on the number of the π -discontinuity points. The minimum-facet-length itself approaches zero as the following arguments show. It is an open

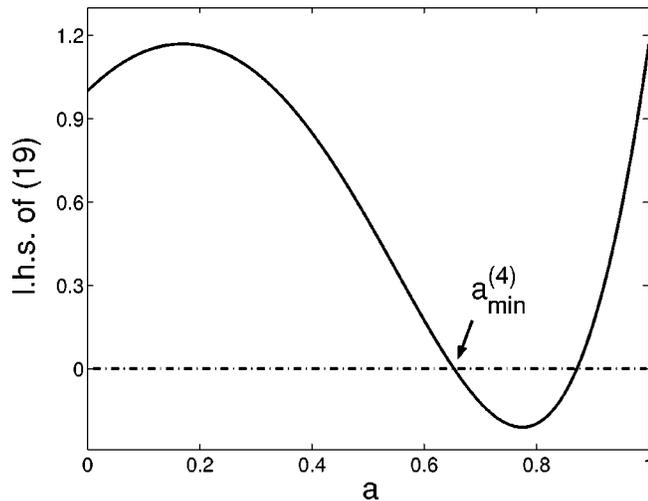


FIG. 9. Plot of the left hand side (LHS) of Eq. (19) as a function of a . The minimum-facet-length $a_{\min}^{(4)}$ is the first intersection of the curve with the a axis.

question what the product is of the minimum-facet-length and the number of the discontinuity points in the limit that this number goes to infinity.

In this limiting case, we consider the scattering problem with a periodic potential. Mathematically, we are looking for solution of the equation

$$\begin{aligned} u_{xx} - \lambda^2 u &= \theta(x)u, \\ \theta(x) &= \theta(x+4a). \end{aligned} \quad (20)$$

Let us assume that $\theta(x) = -1$ for $0 < x < 2a$, and $\theta(x) = 1$ for $2a < x < 4a$.

The general solution of Eq. (20) is of the form²⁵

$$\begin{aligned} u &= e^{iKx} \varphi(x), \\ \varphi(x) &= \varphi(x+4a), \end{aligned} \quad (21)$$

with K satisfying $K4a = 2n\pi$, ($n = 0, \pm 1, \pm 2, \dots$). Substituting Eq. (21) into Eq. (20), we are left with an ordinary differential equation in φ . The solution is described by

$$\varphi = \begin{cases} Ae^{i(\kappa_+ - K)x} + Be^{-i(\kappa_+ + K)x}, & 0 < x < 2a, \\ Ce^{i(\kappa_- - K)x} + De^{-i(\kappa_- + K)x}, & 2a < x < 4a, \end{cases} \quad (22)$$

where $\kappa_{\pm} = \sqrt{\lambda^2 \pm 1}$. The coefficients A, B, C , and D are obtained from continuity and periodicity conditions. With the same argument as before we get the condition

$$\cos a \cosh a = \cos K4a$$

which gives $a_{\min}^{(\infty)} = 0$. The calculation of the minimum-facet-length for infinitely many discontinuity points also tells us that arrays of $0-\pi$ junctions forming a loop or annular junctions containing π -discontinuity points have zero minimum-facet-length.

VI. CONCLUSIONS

We have discussed the existence of static semifluxons using phase plane analysis in one- and two-corner junctions. We have obtained the critical value of the applied bias current γ above which static semifluxons are not present. By phase-plane analysis we have also shown how to construct solutions with humps. We have not discussed the stability of these solutions. We plan to do so in the future.

For two-corner junctions, the exact relation between the magnetic flux of semifluxons and the facet length has been derived. There is a minimum-facet-length for a semivortex-antisemivortex state at $\gamma = 0$.

For multicorner junctions, the minimum-facet-length of the antiferromagnetically ordered semifluxon state is deter-

mined by a stability analysis of the trivial state. The minimum-facet-length is zero for a periodic junction. A similar argumentation as we used to calculate the minimum-facet length in case of infinitely many corner junctions, shows that annular junctions with discontinuity points have zero minimum-facet-length.

Very recently, Zenchuk and Goldobin²⁶ considered the problem of calculating the minimum-facet-length for Josephson junctions with several discontinuity points (Sec. V in this article). They use a different method in considering the case

of an odd number and even but large number of discontinuity points. In addition, they calculated that $a_{\min}^{(N)} \sim 1/\sqrt{N}$ for N even and $N \rightarrow \infty$.

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