Instability of a lattice semifluxon in a current-biased 0-π array of Josephson junctions

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We consider a one-dimensional parallel biased array of small Josephson junctions with a discontinuity point characterized by a phase jump of π in the phase difference. The system is described by a spatially nonautonomous discrete sine-Gordon equation. It is shown that in the infinitely long case there is a semifluxon spontaneously generated attached to the discontinuity point. Comparing the configurations of the semifluxon, we find an energy barrier similar to the Peierls-Nabarro barrier. We calculate numerically the minimum bias current density to overcome this barrier which is a function of the lattice spacing. It is found that the minimum bias current is the critical current for the existence of static lattice semifluxons. For bias current density above the minimum value, the semifluxon changes the polarity and releases 2π fluxons. An analytical approximation to the critical current as a function of the lattice spacing is presented.

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The non-s-wave superconductivity of high-Tc superconductors opens the possibility of an intrinsic π phase shift in a Josephson junction, or in other words an effective negative phase shift in

\[ \phi_n = \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{a^2} = -\sin[\phi_n + \theta(n)] - \alpha \phi_n + \gamma, \]

where 1 \( \leq n \leq 2N \), 2N is the number of junctions/lattices and \( \phi_n \) is the phase difference of the \( n \)th junction. In the simulations we take \( N = \max\{10a,50\} \). A time derivative of \( \theta \) is denoted by \( \dot{\theta} \). Equation (1) is in nondimensionalized form. Parameter \( \alpha \) is the positive damping coefficient related to tunneling of electrons across the junctions and \( \gamma \geq 0 \) is the bias current density. The function \( \theta(n) \) is defined as

\[ \theta(n) = \begin{cases} 0, & 1 \leq n \leq N \\ \pi, & N < n \leq 2N, \end{cases} \]

and to denote the phase jump of \( \pi \) in the phase difference.

In the limit \( a \to 0 \) Eq. (1) corresponds to the differential equation for a long 0-π Josephson junction. Thus, for an array of Josephson junctions with small value of \( a \) one might expect a dynamical behavior similar to that of the continuum limit. It means we can predict the minimum bias current for a semifluxon to release \( 2\pi \) fluxons in this case. The semifluxon admitted by Eq. (1) for \( \gamma = 0 \) in the limits \( a \to 0 \) is given by

\[ \phi_n^0 = \begin{cases} 4 \arctan \exp[x(n)-x_0], & x(n) < 0 \\ 4 \arctan \exp[x(n)+x_0]-\pi, & x(n) \geq 0, \end{cases} \]

where \( x(n) = (n-N)a \) and \( x_0 = \ln(\sqrt{2}+1) \).

The fabrication of the junction as well as the numerics is, of course, limited to finite \( N \). One then deals with boundary conditions. A reasonable choice is to take a boundary condition representing the way in which the applied magnetic field \( \hbar \) enters the system,

\[ \frac{\phi_2 - \phi_1}{a} = \frac{\phi_{2N} - \phi_{2N-1}}{a} = \hbar = 0. \]

A 1D 0-π array of Josephson junctions is described by a discretized version of a spatially nonautonomous sine-Gordon equation.
When \( u(n) \) is constant for all \( n \), Eq. (1) is known also as the Frenkel-Kontorova (FK) model. It has been well-discussed that in the FK model there is a barrier which is called the Peierls-Nabarro (PN) barrier. This PN barrier corresponds to the energy difference of a 2\( \pi \)-kink centered on a lattice site and one centered between two consecutive lattice sites. Because the latter configuration has a minimum energy, one can guess that the spontaneous semifluxon also prefers to sit in the middle between two lattices. The configuration of a lattice semifluxon with low energy is sketched in Fig. 2a.

The other possible configuration [Fig. 2(b)] is unstable, requiring a higher energy. Therefore we observe also an energy barrier similar to the PN barrier.

An interesting question is then to determine how large the energy barrier of the semifluxon is for a given value of the lattice spacing. To simplify the calculation, we compute the energy barrier in terms of the applied bias current \( g \).

We have calculated numerically the minimum bias current to overcome the energy barrier as a function of \( a \). At this value of \( g \), the lattice configuration is given in Fig. 2b.

This result shows that a lattice semifluxon can move along the junction with maximum displacement half of the lattice spacing. Another interesting result is that for an applied bias current \( g \) larger than this minimum value the static lattice semifluxon does not exist. The semifluxon becomes dynamic and changes its polarity into a lattice \( \pi \)-antikink and release an antifluxon. This means the energy barrier is equal to the critical current for the existence of a static lattice semifluxon \( g_c \). A sketch of the transition is presented in Fig. 3. As long as \( g \) is larger than \( g_c \), the process repeats itself. One difference between the moving fluxons of the array and the long 0-\( \pi \) Josephson junction is that the moving lattice fluxons emit radiation at the tails because of the discreteness effect.

The plot of \( g_c \) as a function of the lattice spacing \( a \) is presented in Fig. 4.

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The plot of \( g_c \) as a function of the lattice spacing \( a \) is presented in Fig. 4.
In the case \( a \to 0 \), \( \gamma_c \) has been calculated as \( 2/\pi. \) For \( a \ll 1 \), the critical bias current is close to this value which means that there is a similarity between a long 0-\( \pi \) Josephson junction and a weakly discrete Josephson transmission line. From \( a = 1 \) to approximately \( a = 5 \) there is a significant increment in the value of the critical current. For \( a > 5 \) the change is not so drastic anymore which means that the semifluxon of Eq. (1) ceases to distinguish the difference of the lattice spacings in a very strongly discrete 0-\( \pi \) Josephson junction. In the limit \( a \to \infty \), \( \gamma_c \to 1 \), as will be shown later. This result is also similar to the PN barrier of a lattice 2\( \pi \)-kink of the ordinary discrete sine-Gordon equation where the energy barrier shows a rapid increase near the value \( a = 1 \). The transition marks the difference between the values of \( a \) where the discreteness effects are negligible and values of \( a \) where the kink dynamics are dominated by the discreteness. 11

Next, we will derive an analytical approximation to the critical bias current as a function of \( a \) in two different limiting cases, i.e., \( a \to 0 \) and \( a \to \infty \). Since the critical bias current corresponds to the disappearance of the stable static semifluxon, to determine \( \gamma_c \) we only consider the time-independent equation of Eq. (1).

In the limit of \( a \to \infty \), the equation is given by

\[
\sin[\phi_n + \theta(n)] = \gamma, \tag{5}
\]

where the stable semifluxon (mod 2\( \pi \)) is

\[
\phi_n = \begin{cases} 
\arcsin \gamma, & \text{for } n < N \\
\pi + \arcsin \gamma, & \text{for } n \geq N.
\end{cases} \tag{6}
\]

When \( \gamma > 1 \), there are no solutions to Eq. (5), implying that \( \gamma_c \to 1 \) when \( a \to \infty \).

Analytical calculation of \( \gamma_c \) as a function of \( a \) for \( a \ll 1 \) will be done by considering a continuum approximation of Eq. (1). We will derive a continuum approximation in the region \( x < 0 \) and \( x > 0 \) separately. For simplicity, consider only the discrete sine-Gordon equation for \( x < 0 \)

\[
1/a^2(\phi_{n-1} - 2 \phi_n + \phi_{n+1}) = \sin(\phi_n) - \gamma. \tag{7}
\]

Expanding the left-hand side of the above equation and setting \( \phi_n = \phi \), one obtains

\[
L_A[\partial_x^2 \phi] = \sin \phi - \gamma, \tag{8}
\]

where

\[
L_A = 1 + a^2 \partial_x^2 + a^4 \partial_x^4 + \cdots .
\]

Considering that \( \phi_c \) is a slowly decaying solution and \( a \) is small, one can take the following formal expansion12

\[
L_A^{-1} = 1 - a^2 \frac{\partial_x^2}{12} + O(a^4).
\]

Applying \( L_A^{-1} \) to Eq. (8), we obtain a continuum approximation of Eq. (7) [up to \( O(a^4) \)]

\[
\phi_{xx} = \sin \phi - \gamma \frac{a^2}{12} \left[ \phi_{xx} \cos \phi - \phi_x^2 \sin \phi \right]. \tag{9}
\]

Doing the same procedure to the discrete sine-Gordon equation for \( x > 0 \), an approximate equation of Eq. (1) is then given by

\[
\phi_{xx} = \sin[\phi + \theta(x)] - \gamma \frac{a^2}{12} \left[ \phi_{xx} \cos \phi + \theta(x) \right] - \phi_x^2 \sin[\phi + \theta(x)] , \tag{10}
\]

where \( \theta(x) \) is defined similar to Eq. (2), i.e.,

\[
\theta(x) = \begin{cases} 
0, & x < 0 \\
\pi, & x \geq 0.
\end{cases}
\]

The boundary conditions at the discontinuity point \( x = 0 \) are given by6-8

\[
\lim_{x \to 0} \phi = \lim_{x \to 0} \phi_+ , \quad \lim_{x \to 0} \phi_x = \lim_{x \to 0} \phi_x ^+ .
\]

The approximate first integral of Eq. (10) is

\[
\frac{1}{2} \phi_x^2 = \begin{cases} 
\cos \phi_- - \cos \phi - (\phi - \phi_-) & x < 0 \\
\cos \phi - \cos \phi_+ - (\phi - \phi_+) & \phi_x^2 \cos \phi , \quad x \geq 0,
\end{cases}
\]

Next, it is natural to expand all the quantities as

\[
\phi = \phi_0 + a^2 \phi_1
\]

and

\[
\gamma = \gamma_0 + a^2 \gamma_1
\]

from which \( \gamma_c = \gamma_c^0 + a^2 \gamma_c^1 \). Substituting the expansions to Eqs. (10) and (11) for \( O(a^2) \) we obtain the following equations:

\[
\phi_{xx} = \sin \phi - \gamma \frac{a^2}{12} \left[ \phi_{xx} \cos \phi - \phi_x^2 \sin \phi \right].
\]

\[
\phi_{xx} = \sin[\phi + \theta(x)] - \gamma \frac{a^2}{12} \left[ \phi_{xx} \cos \phi + \theta(x) \right] - \phi_x^2 \sin[\phi + \theta(x)].
\]
For $i.e.$, the appearance of the zero eigenvalue of the semifluxon is satisfied when

$$\phi_x^i(0) = 0, \quad \phi_y^i(0) = \frac{1}{12} \arcsin \left( \frac{1}{2} \right),$$

where $\phi^i = \lim_{x \to +\infty} \phi^i$. For $O(1)$ the equations are Eqs. (10) and (11) with $\phi = \phi^0$ and $a = 0$. It is easy to find out that

$$\phi^0 = \arcsin \frac{\phi}{\phi^0} - \pi. \quad \text{We also obtain that} \quad \phi^0 = \phi^0/\sqrt{1 - \phi^2}.$$  

It is shown in Refs. 6–8 that the critical current corresponds to the condition that

$$\lim_{x \to 0} \phi_x^i = \lim_{x \to 0} \phi_x^i \quad \text{or} \quad \lim_{x \to 0} \phi_x^i = \lim_{x \to 0} \phi_x^i, \quad i = 0, 1, \quad (14)$$

i.e., the appearance of the zero eigenvalue of the semifluxon. For $a = 0$ it has been calculated$^6$–$^8$ that the condition is satisfied when

$$\phi^0(0) = \pi, \quad \phi_x^0(0)^2 = 2 \left[ \sqrt{1 - \gamma^2} - 1 + \gamma^0 \arcsin \left( \gamma^0 \right) \right]$$

from which the critical bias current of $O(1)$ is derived as

$$\gamma^0 = \frac{2}{\pi}.$$  

Next we will calculate $\gamma^i$. From Eq. (14), we obtain

$$\phi^i(0) = -\frac{\gamma^0}{12}. \quad \text{Using the continuity condition that} \quad \lim_{x \to 0} \phi_x^i = \lim_{x \to 0} \phi_x^i, \quad \text{from Eq. (13) we obtain}$$

$$\gamma^i = \frac{\sqrt{1 - \frac{4}{\pi^2} \pi - 2 \arcsin \left( \frac{2}{\pi} \right)}}{3 \pi^2} \approx 0.02234. \quad (15)$$

It is shown in Fig. 4 that the approximate function has a good agreement with the numerical result.

To conclude, we have presented some numerical results on the lattice array of 0–$\pi$ Josephson junctions. It is shown that the array has a spontaneously generated semifluxon attached to the discontinuity point. We have observed that the semifluxon can move along the junctions because of the two possible configurations of the semifluxon. We also have computed the energy difference of the two configurations that is similar to the PN barrier as a function of the lattice spacing. It turns out that the energy barrier corresponds to the critical bias current of the existence of the static lattice semifluxon. An approximate function of the critical current has been calculated as well in two limiting cases, i.e., $a \to 0$ and $a \to \infty$. Since the critical bias current increases when the lattice spacing increases, lattice 0–$\pi$ junctions enlarge the possibility of utilizing the semifluxons in memory cells or logic devices.

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