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Abstract. In this paper we consider a PT-symmetry multi dimer system, which consists of a collection of PT-symmetry dimers, under the influence of a linear potential. The stability of the system is analyzed using a perturbation method because the solution to review is the case of weak binding constraints. Keywords: pt-symmetry, perturbation method, stability

1. Introduction

In the context of applications in the optical field, PT-symmetry system describes the propagation of an optical beam on 2 coupled waveguides where in a single waveguide optics experiences gain and in other waveguide experiences loss. Each intensity of gain/loss has an equivalent value (see reference [1,2] to refer to some related experiments). The two waveguides are also known as the dimer. Research on the propagation of optical rays on this waveguide itself plays an important role in the development of optical-based communication technologies in the future.

In this article, the study of linear versions of the PT-symmetry dimer model discussed in reference [3,4] will be developed for the case multi dimer under linear potential influence. The method used is the perturbation method because the solution to review is the case of weak binding constraints.

Parity-Time symmetry, abbreviated as PT-symmetry, is a fundamental simetry operation in physics [5]. P is a linear operator that reflects parity (position and momentum), that is

$$\mathrm{P}: \hat{x} \to -\hat{x}\ ,\ p \to -p\ , \tag{1}$$

whereas T is an anti-linear operator that reflects time and complex conjugations, i.e

$$T: t \to -t,\ \ i \to -i\ . \tag{2}$$

Interestingly, there are some systems that do not meet the symmetry P and T separately, but meet a combination of symmetries P and T . This is then called the PT -symmetry system.

2. PT-Symmetry Multi Dimer

By expanding PT-Symmetry dimer to be a system, it will consists of a collection of PT-symmetry dimers bound by an inter and between dimers. This system is then called PT-symmetry multi dimer.

By adding a linear potential, the model of PT-symmetry multi dimer is given by the system:

$$\left.\begin{aligned} i\dot{u}_n &= i\gamma u_n + kv_n + 2\alpha n u_n + \varepsilon v_{n-1}, \\ i\dot{v}_n &= -i\gamma v_n + ku_n + 2\alpha n v_n + \varepsilon u_{n+1}, \end{aligned}\right\} \tag{3}$$

where:

1. $u_n \equiv u_n(t)$ and $v_n \equiv v_n(t)$ is a complex value function that states the amplitude of the optical wave propagating on the waveguides at the time of t > 0 and site $n \in Z$.
2. 'dot' represents the fi rst differential of t.
3. $\gamma > 0$ states the gain loss constant.
4. α is a linear potential constant.
5. k > 0 and $\varepsilon \geq 0$ respectively represents the binding constants inter and between dimers.

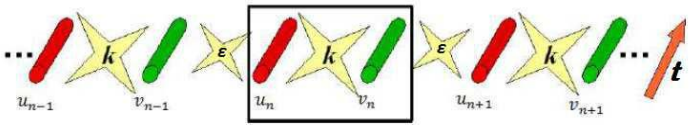

Figure 1. Illustration of PT-symmetry multi dimer in pandu-wave arrangement

3. Analysis

To confirm the nature of PT-symmetry owned by the system (7), it will be shown that the system is invariant of the following transformations:

$$(u_n, v_n, \dot{u}_n, \dot{v}_n, t, i, n) \rightarrow (v_n, u_n, -\dot{v}_n, -\dot{u}_n, -t, -i, -n). \tag{4}$$

By using transformation (8) to system (7) and then assuming n = −m in equation, the system (7) will be returned, so that the system is equivalent to this following system:

$$\left.\begin{aligned} \dot{U}_n &= i\tilde{\gamma} U_n + V_n + 2\tilde{\alpha} n U_n + \tilde{\varepsilon} V_{n-1}, \\ i\dot{V}_n &= -i\tilde{\gamma} V_n + U_n + 2\tilde{\alpha} n V_n + \tilde{\varepsilon} U_{n+1}. \end{aligned}\right\} \tag{5}$$

Where $U_n \equiv U_n(\tau)$, $V_n \equiv V_n(\tau)$, $\tilde{\gamma} = \frac{\gamma}{k}$, $\tilde{\alpha} = \frac{\alpha}{k}$, $\tilde{\varepsilon} = \frac{\varepsilon}{k}$.

Analysis Stability.

Therefore, without reducing generality, by using k=1, the following discussion will consider the following Pt-symmetry multi dimer system:

$$\left.\begin{aligned} i\dot{u}_n &= -i\gamma u_n n + v_n + 2\alpha n u_n + \varepsilon v_{n-1}, \\ i\dot{v}_n &= -i\gamma v_n n + u_n + 2\alpha n v_n + \varepsilon u_{n+1}. \end{aligned}\right\} \tag{6}$$

The PT-symmetry multi dimer system given by equation (10) is a first-order linear differential equation system with dependant variable $u_n(t)$ and $v_n(t)$ which have complex values. Therefore the solution of the system can be written in the form:

$$u_n(t) = p_n e^{i\lambda t} + \bar{q}_n e^{-i\bar{\lambda} t},\ v_n(t) = r_n e^{i\lambda t} + \bar{s}_n e^{-i\bar{\lambda} t}. \quad (7)$$

After that, by substituting the solution (11) to our system, and as $e^{i\lambda t}$ and $e^{-i\bar{\lambda} t}$ are linearly independent, then we obtain a new system consists of 4 equations as follow:

$$\left.\begin{aligned}
&(-i\gamma + 2\alpha n)\,\bar{s}_n + \varepsilon\,\bar{q}_{n+1} + \bar{q}_n = \bar{\lambda}\,\bar{s}_n,\\
&(-i\gamma + 2\alpha n)\,r_n - \varepsilon\,p_{n+1} - p_n = \lambda\, r_n,\\
&(i\gamma + 2\alpha n)\,\bar{q}_n + \varepsilon\,\bar{s}_{n-1} + \bar{s}_n = \bar{\lambda}\,\bar{q}_n,\\
&(-i\gamma + 2\alpha n)\,p_n + \varepsilon\, r_{n-1} + r_n = \lambda\, p_n.
\end{aligned}\right\} \quad (8)$$

System (12) consist of 2 pairs uncoupled system where both systems are eigen values problem for system (10), here eigen values notated as $i\gamma$. As for that hen the stationary solution of the system (10) is stable if and only if Im (λ) = 0. Let consider just only one of the uncouple system, at n + 1, n and n − 1, it is obtained the following system equation:

$$\left.\begin{aligned}
&-(i\gamma + 2\alpha(n-1))p_{n-1} + \varepsilon r_{n-2} - r_{n-1} = \lambda p_{n-1},\\
&-(i\gamma + 2\alpha n)p_n + \varepsilon r_{n-1} - r_n = \lambda p_n,\\
&-(i\gamma - 2\alpha(n+1))p_{n+1} + \varepsilon r_n - r_{n+!} = \lambda p_{n+1},\\
&-p_{n-1} + \varepsilon p_n + (i\gamma - 2\alpha(n-1))r_{n-1} = \lambda r_{n-1},\\
&-p_n + \varepsilon p_{n+1} + (i\gamma - 2\alpha n)r_n = \lambda r_n,\\
&-p_{n+1} + \varepsilon p_{n+2} + (i\gamma - 2\alpha(n+1)r_{n+1} = \lambda r_{n+1},
\end{aligned}\right\} \quad (9)$$

with boundary conditions are $r_{n-2} = 0$ and $p_{n+2} = 0.$

The system of linear equations above can be reduced to one linear equation in form p_n as equation below:

$$F(\alpha, \gamma, \varepsilon, n, \lambda)p_n = 0. \quad (10)$$

Analysis stability of the PT-symmetry multi dimer system (10) for $\varepsilon \ll 1$ examined by finishing the eigenvalue problem (12) using the perturbation expansion method. Suppose asymptotic expansion for eigenvectors p_n and eigenvalues λ can be written in the form:

$$\begin{aligned}
&p_n = p_n^{(0)} + \varepsilon p_n^{(1)} + \varepsilon^2 p_n^{(2)} + O(\varepsilon^3),\\
&\lambda = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + O(\varepsilon^3).
\end{aligned} \quad (11)$$

By substituting asymptotic expansion (15) into equation (14) then collecting the tribes based on the rank ε, the equation is obtained following:

$$\begin{aligned}
&O(1) : C_1 p_n^{(0)} = 0,\\
&O(\varepsilon) : C_1 p_n^{(1)} + C_2 p_n^{(0)} = 0,\\
&O(\varepsilon^2) : C_1 p_n^{(2)} + C_2 p_n^{(1)} + C_3 p_n^{(0)} = 0.
\end{aligned} \quad (12)$$

C_1, C_2, and C_3 obtained by solving system (16), and by that we obtain $\lambda^{(0)}, \lambda^{(1)}$ and $\lambda^{(2)}$. The result of $\lambda^{(0)}$ are below:

$$\lambda_1^{(0)} = -\alpha n - 2\alpha + \sqrt{-\gamma^2 + 1},$$
$$\lambda_2^{(0)} = -2\alpha n - 2\alpha - \sqrt{-\gamma^2 + 1},$$
$$\lambda_3^{(0)} = -2\alpha n + \sqrt{-\gamma^2 + 1},$$

$$\lambda_4^{(0)} = -2\alpha n - \sqrt{-\gamma^2 + 1},$$
$$\lambda_5^{(0)} = -2\alpha n + 2\alpha + \sqrt{-\gamma^2 + 1},$$
$$\lambda_6^{(0)} = -2\alpha n + 2\alpha - \sqrt{-\gamma^2 + 1}, \quad (13)$$

and for $\lambda^{(1)}$,

$\lambda^{(1)} = 0.$

Based on these results, the criteria for stability of the solution can be formulated in the following points:

1. When ε = 0, the stability of the solution is determined from the eigen value of $\lambda_k^{(0)}$ who given by equation (13). Next, we will prove that Im ($\lambda_k^{(0)}$) = 0 for every k = 1, 2, ..., 6. Note that α ∈ R and n ∈ Z. Thus the stability of the solution is enough to be seen from $\sqrt{-\gamma^2+1}$. It is clear that $\sqrt{-\gamma^2+1}$ ∈ R if and only if γ ≤ 1. But because γ has been set at the beginning it is positive, then $\sqrt{-\gamma^2+1}$ ∈ R if and only if 0 <γ ≤ 1.
2. Note that the eigenvalue is $\lambda_k^{(2)}$ has a complex value where $\mathrm{Im}(\lambda_k^{(2)}) \neq 0.$ Thus, the stationary solution from the system (10) behaves unstable for 0 <ε ≪ 1.

4. Conclusion

In this journal, the analysis of the stability of the solution of PT-symmetry multi dimer has been discussed. The system has form from expanding and collecting system of PT- symmetry dimer, bound by an inter and between dimers.

Analysis stability of the PT-symmetry multi dimer system for ε ≪ 1 examined by finishing the eigenvalue problem using the perturbation expansion method. Furthermore, the results of stability analysis are as follows:

1. When ε= 0, the stationary solution of the system behaves stable if and only if 0 <γ ≤ 1.
2. At 0 <ε ≪ 1, stationary solutions from the system behave unstable.

5. Acknowledgements

This research is supported by Research and Comuunity Services Dharma Andalas University, Padang, Indonesia.

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