Travelling Waves in Nonlinear Magnetic Metamaterials

J. Diblík, M. Fečkan, M. Pospíšil, V.M. Rothos, and H. Susanto

Abstract In this article, a model of one-dimensional metamaterial formed by a discrete array of nonlinear resonators is considered. The existence and uniqueness results of periodic and asymptotic travelling waves of the system are presented. The existence and the stability of asymptotic waves are also computed and discussed numerically.

J. Diblík
Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 3058/10, 616 00 Brno, Czech Republic
e-mail: diblik@feec.vutbr.cz

M. Fečkan (✉)
Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia

Mathematical Institute of Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
e-mail: Michal.Feckan@fmph.uniba.sk

M. Pospíšil
Centre for Research and Utilization of Renewable Energy, Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 3058/10, 616 00 Brno, Czech Republic
e-mail: pospisilm@feec.vutbr.cz

V.M. Rothos
Department of Mathematics, Physics and Computational Sciences, Mathematics Division, Faculty of Engineering, Aristotle University of Thessaloniki, Thessaloniki GR54124, Greece
e-mail: rothos@auth.gr

H. Susanto
School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK
e-mail: Hadi.Susanto@nottingham.ac.uk

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1 Introduction

Metamaterials are artificial materials that are engineered to have properties that may not be found in nature. The modification is achieved by composing inhomogeneities, i.e. structural rather than chemical, to create desirable effective behaviour. Primarily the study is on engineering the refractive index of a material. Since the proposal [21], a new paradigm in electromagnetism has emerged due to these new types of artificial composites, including cloaking devices [22] (see also [18] for a review of recent results for electromagnetic manipulation enabled by metamaterials). The canonical constituent components, or meta-atoms from which metamaterials are fashioned are the split ring resonator, which consists of an inductive metallic ring with a gap to provide capacitance, or its U-shaped modifications.

Previous studies of metamaterials were focused on the linear properties of the medium during wave propagation, in which case magnetic permeability and material permittivity are non-dependent on the intensity of the electromagnetic field. When nonlinearity is present in metamaterials due to either employing a nonlinear host medium [20] or by engineering the elements of a metamaterial with a nonlinear component [13], nontrivial properties and behaviours can be present, i.e. dynamic tunable systems and active artificial media, such as materials that are compressible by magnetic field [15] and devices that can switch from a normal flat mirror to a focusing/defocusing one without changing shape [24] (see also [23] for a review).

The propagation of electromagnetic waves in metamaterials has been discussed and modelled by several types of nonlinear equations, such as a nonlinear Klein-Gordon equation [19], coupled short-pulse equation [28], higher-order nonlinear Schrödinger equations [27] and coupled Klein-Gordon equations [14, 17]. The latter equations are of the form

$$\begin{aligned} \frac{dq_n}{dt} &= i_n, \qquad n \in \mathbb{Z}, \\ \frac{d}{dt}\left(\lambda i_{n-1} - i_n + \lambda i_{n+1}\right) &= \gamma i_n - f(t) + \varphi(q_n), \end{aligned} \tag{1}$$

with loss coefficient γ, coupling parameter λ, external forcing f and nonlinear function φ. The coupled equations model the dynamics of the capacitor charge of split-ring resonators or U-type elements in magnetic metamaterials [14, 17]. Due to the nonlinearity, localization of electromagnetic waves is possible. For Eq. (1), localized excitations in the form of discrete breathers and domain walls were observed. In this article, we investigate the existence of travelling periodic solutions of system (1) with periodic forcing, and the bifurcation of periodic solutions and asymptotic waves of (1) with small γ, λ and f. We consider in particular the case when the external force also varies in space.

The present paper is organized as follows. In Sect. 2 we are looking for a periodic travelling wave when its amplitude is limited by the magnitude of the forcing. In Sect. 3, we suppose that an unperturbed system (see (21)) has a periodic solution, and we use the subharmonic Melnikov bifurcation method to find conditions under which this periodic solution persists. Homoclinic Melnikov bifurcation method is used in Sect. 4. We obtain large solutions in Sects. 3 and 4 under small perturbations, while in Sect. 2 we obtain a small solution under small perturbation. The main difference is the resonance property, i.e. Sect. 2 is dealing with non-resonances (see condition (9)), while in Sect. 3 there is a resonance assumption (H2). In Sect. 5 we solve the governing equations numerically, in particular for the asymptotic waves. Comparisons with the analytical results are presented where we obtain good agreement.

2 Existence Results on Periodic Solutions

In this section, we study the existence of periodic solutions of system (1) forced by a travelling wave field when the amplitude of periodic solutions is proportional to the amplitude of forcing. Thus, we consider the equivalent equation

$$\lambda\ddot{q}_{n+1} - \ddot{q}_n + \lambda\ddot{q}_{n-1} = \gamma\dot{q}_n + \varphi(q_n) + f\cos(\omega t + pn), \tag{2}$$

where as above $\gamma \geq 0$ is the dissipative loss of the medium, $\lambda \in \mathbb{R}$ is the coupling constant between nearest neighbor resonators, $\omega > 0$ is the external driving frequency, $f \neq 0$ is the amplitude of the external force, $p \neq 0$ is the wavenumber of the travelling wave field and φ is the nonlinearity of the magnetic material, which is normally assumed to be of Kerr-type [17]. Therefore, we assume that $\varphi(z)$ is an odd analytic function in its variable z with radius of convergence $\rho > 0$ such that

$$\mathrm{D}\varphi(0) = 0. \tag{3}$$

Seeking for waves travelling in the same direction as the external drive, we take $q_n(t) = U(z)$, $z = \omega t + pn$, in (2) with $U(z + \pi) = -U(z)$ to obtain

$$\omega^2(\lambda U''(z+p) - U''(z) + \lambda U''(z-p)) = \gamma\omega U'(z) + \varphi(U(z)) + f\cos z. \tag{4}$$

Note that $p \in \mathbb{R}\backslash\{0\}$. Considering the case $p = -\pi$, one will obtain alternating charges between the nearest-neighbor resonators as $f\cos(\omega t + pn) = (-1)^n f\cos\omega t$.

We take Banach spaces

$$W := \left\{ U \in C(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k \in \mathbb{Z}} d_k \, \mathrm{e}^{k\imath z}, \ \sum_{k \in \mathbb{Z}} |d_k| < \infty \right\},$$

$$X := \left\{ U \in C(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k \in \mathbb{Z}} c_k \, \mathrm{e}^{(2k+1)\imath z}, \ \sum_{k \in \mathbb{Z}} |c_k| < \infty \right\},$$

$$Y := \left\{ U \in C^1(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k \in \mathbb{Z}} c_k \, \mathrm{e}^{(2k+1)\imath z}, \ \sum_{k \in \mathbb{Z}} |2k+1||c_k| < \infty \right\},$$

$$Z := \left\{ U \in C^2(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k \in \mathbb{Z}} c_k \, \mathrm{e}^{(2k+1)\imath z}, \ \sum_{k \in \mathbb{Z}} (2k+1)^2 |c_k| < \infty \right\}$$

with the norms

$$\|U\| := \sum_{k \in \mathbb{Z}} |d_k|, \quad \|U\| := \sum_{k \in \mathbb{Z}} |c_k|,$$

$$\|U\|_1 := \sum_{k \in \mathbb{Z}} |2k+1||c_k|, \quad \|U\|_2 := \sum_{k \in \mathbb{Z}} (2k+1)^2 |c_k|,$$

respectively. It is easy to verify that $Z \circlearrowleft Y \circlearrowleft X$ are compact embeddings, $X \subset W$ and $\|U\| \le \|U\|_1 \ \forall U \in Y$, $\|U\|_1 \le \|U\|_2 \ \forall U \in Z$.

The following lemma is clear.

Lemma 1. *If $U_1, U_2 \in W$ then $U_1 U_2 \in W$ and $\|U_1 U_2\| \le \|U_1\| \|U_2\|$. For each $k \in \mathbb{N}$, if $U_1, U_2, \ldots, U_{2k+1} \in X$ then $U_1 U_2 \ldots U_{2k+1} \in X$.*

By setting

$$\mathcal{K}U := \omega^2 (\lambda U''(z+p) - U''(z) + \lambda U''(z-p)) - \gamma \omega U'(z),$$

$$\mathcal{F}(U, f) := \varphi(U) + f \cos z,$$

Eq. (4) has the form

$$\mathcal{K}U = \mathcal{F}(U, f).$$

Denote $B(\rho) := \{U \in Y \mid \|U\|_1 < \rho\}$.

We have the next result.

Lemma 2. *Function $\mathcal{F} : B(\rho) \times \mathbb{R} \to X$ fulfils*

$$\|\mathcal{F}(U, f)\| \le \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi(0)|}{k!} \|U\|_1^k + |f|, \tag{5}$$

$$\|\mathcal{F}(U_1, f) - \mathcal{F}(U_2, f)\| \leq \|U_1 - U_2\|_1 \sum_{k=3}^{\infty} \frac{|D^k\varphi(0)|}{k!} \sum_{j=0}^{k-1} \|U\|_1^j \|U_2\|_1^{k-j-1}, \quad (6)$$

$$\|\mathcal{F}(U, f_1) - \mathcal{F}(U, f_2)\| \leq |f_1 - f_2| \quad (7)$$

for any $U, U_1, U_2 \in B(\rho) \subset Y$ *and* $f, f_1, f_2 \in \mathbb{R}$.

Proof. Since $\cos z = \frac{e^{\imath z} + e^{-\imath z}}{2}$, we get $\|\cos z\| = 1$ and (7) easily follows. By applying Lemma 1 and estimating the Taylor series

$$\|\varphi(U)\| \leq \sum_{k=3}^{\infty} \frac{|D^k\varphi(0)|}{k!} \|U\|_1^k$$

we obtain (5). Using similar estimation for $\|\varphi(U_1) - \varphi(U_2)\|$ and the identity

$$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1})$$

for any $a, b \in \mathbb{C}$, we arrive at (6).

Now, if $U \in Z$ with $U(z) = \sum_{k\in\mathbb{Z}} c_k\, e^{(2k+1)\imath z}$ then

$$\mathcal{K}U(z) = \sum_{k\in\mathbb{Z}} \left[-\omega^2(2k+1)^2\,(2\lambda\cos(2k+1)p - 1) - \imath\gamma\omega(2k+1)\right] c_k\, e^{(2k+1)\imath z} \quad (8)$$

and so $\mathcal{K} \in L(Z, X)$ with

$$\|\mathcal{K}\|_{L(Z,X)} \leq \omega^2(1 + 2|\lambda|) + \gamma\omega.$$

If

$$\Theta := \inf_{k\in\mathbb{Z}} \sqrt{\omega^4(2k+1)^2\,(2\lambda\cos(2k+1)p - 1)^2 + \gamma^2\omega^2} > 0 \quad (9)$$

is a constant depending on γ, λ, ω and p, then we also have $\mathcal{K}^{-1} \in L(X, Z) \subset L(X, Y) \subset L(X)$. So $\mathcal{K}^{-1} : X \to Y$ is compact such that

$$\|\mathcal{K}^{-1}\|_{L(X,Y)} \leq \frac{1}{\Theta}. \quad (10)$$

Now we can prove the following existence results on (4) when all parameters except f are fixed.

Theorem 1. *Assume* (9) *along with*

$$|f| < |f_l| \quad (11)$$

for f_l satisfying

$$A(r) := \Theta r - \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi(0)|}{k!} r^k = |f_l|, \tag{12}$$

$$\mathrm{D}A(r) = 0 \tag{13}$$

for some $r \in (0, \rho)$. Then Eq. (4) has a unique solution $U(f) \in \overline{B(\rho_f)}$ in a closed ball where $\rho_f < \rho$ is the smallest positive root of $A(r) = |f|$. Moreover, $U(f)$ can be approximated by an iteration process. Finally, it holds

$$\|U(f_1) - U(f_2)\|_1 \leq \frac{|f_1 - f_2|}{\Theta - \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi(0)|}{(k-1)!} \rho_{\max\{|f_1|,|f_2|\}}^{k-1}} \tag{14}$$

for any $f_1, f_2 \in \mathbb{R}$ satisfying (11).

Proof. We rewrite (4) as a parameterized fixed point problem

$$U = \mathcal{R}(U, f) := \mathcal{K}^{-1} \mathcal{F}(U, f)$$

in $B(\rho) \subset Y$. We already know that $\mathcal{R} : B(\rho) \times \mathbb{R} \to Y$ is compact, continuous and by (5), (10) such that

$$\|\mathcal{R}(U, f)\|_1 \leq \frac{1}{\Theta} \left(\sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi(0)|}{k!} \|U\|_1^k + |f| \right).$$

Next, if there is $0 < \rho_f < \rho$ such that

$$A(\rho_f) = |f|, \tag{15}$$

then $\mathcal{R}(\cdot, f)$ maps $\overline{B(\rho_f)}$ into itself. So it remains to study (15). In order to find the largest f_l for which (15) has a solution $\rho_{f_l} > 0$, we need to solve $A(r) = |f_l|$ together with (13) for $r \in (0, \rho)$. This implies (11). Note that

$$\pm \mathrm{D}A(r) = \pm \Theta \mp \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi(0)|}{(k-1)!} r^{k-1} > \mathrm{D}A(\rho_{f_l}) = 0$$

for $r \in (0, \rho)$, $\pm r < \pm \rho_{f_l}$, and $\lim_{r \to \rho^-} \mathrm{D}A(r) = -\infty$ (see Sects. 7.21, 7.22 and 7.31 of [26]). Hence ρ_{f_l} is uniquely determined by (13). Moreover, continuity of $A(r)$ with $A(0) = 0$, $A(\rho_{f_l}) = |f_l|$ yield that $0 < \rho_f < \rho_{f_l}$ whenever $0 < |f| < |f_l|$, i.e. $\mathrm{D}A(\rho_f) > 0$. So assuming (9), (11) and by (3), we know that (15) has a positive solution $\rho_f < \rho$. We take the smallest one. So $\mathcal{R}(\cdot, f)$ maps $\overline{B(\rho_f)}$ into itself and, moreover, by (6), (10)

$$\|\mathcal{R}(U_1, f) - \mathcal{R}(U_2, f)\|_1 \leq \frac{\|U_1 - U_2\|_1}{\Theta} \sum_{k=3}^{\infty} \frac{|D^k \varphi(0)|}{(k-1)!} \rho_f^{k-1}$$

for any $U_1, U_2 \in \overline{B(\rho_f)}$. Hence, $\mathcal{R}(\cdot, f)$ is a contraction on $\overline{B(\rho_f)}$ with a contraction constant

$$\frac{1}{\Theta} \sum_{k=3}^{\infty} \frac{|D^k \varphi(0)|}{(k-1)!} \rho_f^{k-1} = \frac{\Theta - DA(\rho_f)}{\Theta} < \frac{\Theta - DA(\rho_{f_l})}{\Theta} = 1.$$

The proof of the existence and uniqueness is finished by the Banach fixed point theorem [2]. Next, let $f_1, f_2 \in \mathbb{R}$ satisfy (11), then $U(f_i) \in \overline{B(\rho_{f_i})} \subset \overline{B(\rho_{f_3})}$ for $i = 1, 2$ and $f_3 := \max\{|f_1|, |f_2|\}$. Note f_3 satisfies (11). By (6), (7) and (10), we derive

$$\|U(f_1) - U(f_2)\|_1 = \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_2)\|_1$$
$$\leq \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_1)\|_1 + \|\mathcal{R}(U(f_2), f_1) - \mathcal{R}(U(f_2), f_2)\|_1$$
$$\leq \frac{\|U(f_1) - U(f_2)\|_1}{\Theta} \sum_{k=3}^{\infty} \frac{|D^k \varphi(0)|}{(k-1)!} \rho_{f_3}^{k-1} + \frac{|f_1 - f_2|}{\Theta}$$

which implies (14).

Theorem 2. *Let* (9) *be fulfilled and* f_l *satisfies* (12), (13) *for some* $r \in (0, \rho)$. *Then Eq.* (4) *has a solution* $U(f_l) \in \overline{B(\rho_{f_l})}$ *in a closed ball where* $\rho_{f_l} < \rho$ *is the smallest positive root of* $A(r) = |f_l|$.

Proof. We already know that $\mathcal{R}(\cdot, f_l)$ maps $\overline{B(\rho_{f_l})}$ into itself, and $\mathcal{R}(\cdot, f_l)$ is compact. The Schauder fixed point theorem implies the result.

Remark 1. In the following special cases, (9) holds and we can replace Θ with the corresponding Θ_i in the above considerations:

1. If $\gamma > 0$ then $\Theta \geq \Theta_1 := \gamma\omega > 0$.
2. If $|\lambda| < 1/2$ then

$$\Theta \geq \inf_{k \in \mathbb{Z}} \omega^2 |2k+1| (1 - 2\lambda \cos(2k+1)p) \geq \omega^2 (1 - 2|\lambda|) =: \Theta_2 > 0.$$

3. If $|\lambda| > 1/2$, $\mu := \arccos \frac{1}{2\lambda}$, we can apply the identity

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

to obtain

$$2\lambda \cos(2k+1)p - 1 = -4\lambda \sin \frac{(2k+1)p + \mu}{2} \sin \frac{(2k+1)p - \mu}{2}.$$

Next, if $\nu := \frac{\mu}{p} \in 2\mathbb{Z}$ and p has the form $p = \frac{2p_1+1}{p_2}\pi$ with $2p_1 + 1$ and p_2 relatively prime integers (their only common divisor is 1), then we can write

$$|2\lambda \cos(2k+1)p - 1| = 4|\lambda|| \sin K_+(k)|| \sin K_-(k)| \tag{16}$$

where $K_\pm(k) := \frac{(2k+1\pm\nu)(2p_1+1)\pi}{2p_2}$. So it holds

$$K_\pm(k) = \frac{(\text{odd number})\pi}{\text{even number}} \notin \pi\mathbb{Z}$$

for each $k \in \mathbb{Z}$. Moreover, note that $\sin K_\pm(k+p_2) = -\sin K_\pm(k)$ $\forall k \in \mathbb{Z}$. Therefore, it is sufficient to take in (9) inf for k from a set of only $|p_2|$ subsequent integers, i.e.

$$\begin{aligned} \Theta &\geq \inf_{k\in\mathbb{Z}} \omega^2|2k+1||2\lambda\cos(2k+1)p - 1| \\ &\geq 4\omega^2|\lambda| \min_{k=1,\ldots,|p_2|} |\sin K_+(k)||\sin K_-(k)| =: \Theta_3 > 0. \end{aligned} \tag{17}$$

4. If $|\lambda| > 1/2$ and $\nu := \frac{\mu}{p} \in \mathbb{R}\backslash\mathbb{Q}$ for $\mu := \arccos\frac{1}{2\lambda}$, $p = \frac{p_1}{p_2}\pi$ with relatively prime $p_1, p_2 \in \mathbb{Z}\backslash\{0\}$, then we get equality (16) where

$$K_\pm(k) := \frac{(2k+1\pm\nu)p_1\pi}{2p_2} \notin \pi\mathbb{Z} \tag{18}$$

for each $k \in \mathbb{Z}$. Furthermore, $\sin K_\pm(k+p_2) = (-1)^{p_1}\sin K_\pm(k)$ $\forall k \in \mathbb{Z}$. Hence, (17) holds with $K_\pm(k)$ given by (18).

Remark 2. If (3) does not hold, i.e., $\mathrm{D}\varphi(0) \neq 0$ then we include $-\mathrm{D}\varphi(0)U(z)$ into $\mathcal{K}$ to get:

$$\tilde{\mathcal{K}}U := \omega^2(\lambda U''(z+p) - U''(z) + \lambda U''(z-p)) - \gamma\omega U'(z) - \mathrm{D}\varphi(0)U(z).$$

So if $U(z) = \sum_{k\in\mathbb{Z}} c_k\, \mathrm{e}^{(2k+1)\imath z}$ then

$$\tilde{\mathcal{K}}U(z) = \sum_{k\in\mathbb{Z}} \left[-\omega^2(2k+1)^2\,(2\lambda\cos(2k+1)p - 1) - \imath\gamma\omega(2k+1) - \mathrm{D}\varphi(0)\right] c_k\, \mathrm{e}^{(2k+1)\imath z}.$$

Hence by supposing

$$\tilde{\Theta} := \inf_{k\in\mathbb{Z}} \sqrt{\left(\omega^2(2k+1)(2\lambda\cos(2k+1)p - 1) - \frac{\mathrm{D}\varphi(0)}{2k+1}\right)^2 + \gamma^2\omega^2} > 0$$

we get that $\tilde{\mathcal{K}}^{-1} : X \to Y$ is compact such that

$$\|\tilde{\mathcal{K}}^{-1}\|_{L(X,Y)} \leq \frac{1}{\tilde{\Theta}}.$$

So we can follow our method.

3 Bifurcation Results for Periodic Travelling Waves

In this section, we consider (1) with small γ, λ and f. So we consider the system

$$\begin{aligned} \frac{dq_n}{dt} &= i_n, \quad n \in \mathbb{Z} \\ \frac{d}{dt}\left(\varepsilon\lambda i_{n-1} - i_n + \varepsilon\lambda i_{n+1}\right) &= \varepsilon\gamma i_n - \varepsilon h(\omega t + pn) + \varphi(q_n), \end{aligned} \tag{19}$$

for C^2-smooth and 2π-periodic h, $\varphi \in C^2(\mathbb{R}, \mathbb{R})$ and $\omega > 0$, $p \neq 0$, and $\varepsilon \neq 0$ is a small parameter. Equation (19) implies

$$(\varepsilon\lambda \ddot{q}_{n-1} - \ddot{q}_n + \varepsilon\lambda \ddot{q}_{n+1}) = \varepsilon\gamma \dot{q}_n - \varepsilon h(\omega t + pn) + \varphi(q_n). \tag{20}$$

Putting $q_n(t) = U(\omega t + pn)$ for $U \in C^2(\mathbb{R}, \mathbb{R})$ in (20), we get

$$\omega^2 U''(z) + \varphi(U(z)) - \varepsilon\lambda\omega^2 \left(U''(z-p) + U''(z+p)\right) + \varepsilon\gamma\omega U'(z) - \varepsilon h(z) = 0. \tag{21}$$

Now, we suppose

(H1) $U''(z) + \varphi(U(z)) = 0$ has a $\overline{T}$-periodic solution U_0.

Remark 3. Since $U_0(-z + c_0)$ also solves $U''(z) + \varphi(U(z)) = 0$ and there is $z_0 \in \mathbb{R}$ such that $U_0'(z_0) = 0$, we may suppose that $U_0(0) = 0$ and then $U_0(z) = U_0(-z)$.

Then

$$U_\omega(z) := U_0(z/\omega)$$

satisfies $\omega^2 U_\omega''(z) + \varphi(U_\omega(z)) = 0$. Note U_ω is $T_\omega := \overline{T}\omega$-periodic and even. We assume the resonance condition

(H2) $T_\omega = 2\pi \frac{u}{v}$ for $u, v \in \mathbb{N}$.

Now, we follow the standard subharmonic Melnikov method [2, 3, 12] to (21) based on the Lyapunov-Schmidt method, but for reader's convenience we present more details. First, we take Banach spaces

$$X := \left\{U \in C^2(\mathbb{R},\mathbb{R}) \mid U(z+T) = U(z)\ \forall z \in \mathbb{R}\right\},$$
$$Y := \{U \in C(\mathbb{R},\mathbb{R}) \mid U(z+T) = U(z)\ \forall z \in \mathbb{R}\}$$

with the usual maximum norms $\|\cdot\|_2$ and $\|\cdot\|_0$, respectively, where $T := 2\pi u = \overline{T}\omega v$. Then we split

$$Y = Y_1 \oplus Y_2, \quad X = X_1 \oplus X_2$$

with

$$Y_1 := \operatorname{span}[U'_\omega], \quad Y_2 := \left\{U \in Y \mid \int_0^T U(z)U'_\omega(z)dz = 0\right\},$$
$$X_1 := Y_1, \quad X_2 := \left\{U \in X \mid \int_0^T U(z)U'_\omega(z)dz = 0\right\}.$$

Next, we take the projections $P : Y \to Y_1$ and $Q = \mathbb{I} - P : Y \to Y_2$ defined as

$$PU := \frac{\int_0^T U(s)U'_\omega(s)ds}{\int_0^T U'^2_\omega(s)ds} \times U_\omega.$$

Now, we split

$$U(z+\alpha) = U_\omega(z) + V(z), \quad V \in X_2 \tag{22}$$

in (21) to get

$$\begin{aligned}\omega^2 V''(z) + \varphi'(U_\omega(z))V(z) = \varepsilon\lambda\omega^2\left(U''_\omega(z-p) + U''_\omega(z+p)\right) - \varepsilon\gamma\omega U'_\omega(z) + \varepsilon h(z+\alpha)\\ +\varphi(U_\omega(z)) + \varphi'(U_\omega(z))V(z) - \varphi(U_\omega(z) + V(z))\\ +\varepsilon\lambda\omega^2\left(V''(z-p) + V''(z+p)\right) - \varepsilon\gamma\omega V'(z).\end{aligned} \tag{23}$$

Now, we use the following well-known result [3, 12].

Lemma 3. *Equation $\omega^2 V''(z) + \varphi'(U_\omega(z))V(z) = \tilde{Y}(z) \in Y$ has a solution $V \in X$ if and only if $\tilde{Y} \in Y_2$. This solution is unique requiring $V \in X_2$. Moreover, there is a constant $C_\omega > 0$ such that $\|V\|_2 \le C_\omega \|\tilde{Y}\|_0$.*

Lemma 3 allows us to apply the Lyapunov-Schmidt procedure to (23) as follows

$$\omega^2 V''(z) + \varphi'(U_\omega(z))V(z) = QH(z) \tag{24}$$

and

$$PH(z) = 0 \tag{25}$$

for

$$\begin{aligned} H(z) := \varepsilon\lambda\omega^2 \left(U_\omega''(z-p) + U_\omega''(z+p)\right) - \varepsilon\gamma\omega U_\omega'(z) + \varepsilon h(z+\alpha) \\ +\varphi(U_\omega(z)) + \varphi'(U_\omega(z))V(z) - \varphi(U_\omega(z) + V(z)) \\ +\varepsilon\lambda\omega^2 \left(V''(z-p) + V''(z+p)\right) - \varepsilon\gamma\omega V'(z). \end{aligned}$$

Using

$$H(z) := O(\varepsilon) + O\left(\|V^2\|_2\right)$$

together with Lemma 3 and the implicit function theorem [2], we can uniquely solve (24) to get $V(z) = V_\varepsilon(z) = O(\varepsilon)$. Putting V_ε in (25) we get

$$\begin{aligned} \int_0^T \Big\{\lambda\omega^2 \left(U_\omega''(z-p) + U_\omega''(z+p)\right) - \gamma\omega U_\omega'(z) + h(z+\alpha) \\ + \frac{\varphi(U_\omega(z)) + \varphi'(U_\omega(z))V_\varepsilon(z) - \varphi(U_\omega(z) + V_\varepsilon(z))}{\varepsilon} \\ +\lambda\omega^2 \left(V_\varepsilon''(z-p) + V_\varepsilon''(z+p)\right) - \gamma\omega V_\varepsilon'(z)\Big\} U_\omega'(z) dz = 0 \end{aligned} \tag{26}$$

for $\varepsilon \neq 0$ small. Since

$$\begin{aligned} \frac{\varphi(U_\omega(z)) + \varphi'(U_\omega(z))V_\varepsilon(z) - \varphi(U_\omega(z) + V_\varepsilon(z))}{\varepsilon} = O\left(\frac{\|V_\varepsilon\|_2^2}{\varepsilon}\right) = O(\varepsilon), \\ \lambda\omega^2 \left(V_\varepsilon''(z-p) + V_\varepsilon''(z+p)\right) - \gamma\omega V_\varepsilon'(z) = O(\varepsilon), \end{aligned}$$

(26) is equivalent to

$$M^{u/v}(\alpha) + O(\varepsilon) = 0$$

for

$$\begin{aligned} M^{u/v}(\alpha) := \int_0^T \left(\lambda\omega^2 \left(U_\omega''(z-p) + U_\omega''(z+p)\right) - \gamma\omega U_\omega'(z) + h(z+\alpha)\right) U_\omega'(z) dz \\ = \int_0^T \left(-\gamma\omega U_\omega'(z) + h(z+\alpha)\right) U_\omega'(z) dz, \end{aligned} \tag{27}$$

since U_ω is even. So the small coupling parameter $\varepsilon\lambda$ has no influence in the first order Melnikov function.

Summarizing, we obtain the following result.

Theorem 3. *Suppose (H1) and (H2). If there is a simple zero α_0 of a Melnikov function (27), i.e. $M^{u/v}(\alpha_0) = 0$ and $D_\alpha M^{u/v}(\alpha_0) \neq 0$, then there is a $\delta > 0$ such that for any $0 \neq \varepsilon \in (-\delta, \delta)$ there is a unique $2\pi u$-periodic solution $U(z)$ of (21) with*

$$U(z) = U_0\left(\frac{z-\alpha_0}{\omega}\right) + O(\varepsilon).$$

Proof. The proof follows immediately from the implicit function theorem applied to (26).

Note

$$\begin{aligned}M^{u/v}(\alpha) &= \int_0^T \left(-\gamma\omega U_\omega'(z) + h(z+\alpha)\right) U_\omega'(z)dz\\ &= \frac{1}{\omega}\int_0^{\overline{T}\omega v} \left(-\gamma U_0'(z/\omega) + h(z+\alpha)\right) U_0'(z/\omega)dz\\ &= \int_0^{\overline{T}v} \left(-\gamma U_0'(z) + h(\omega z+\alpha)\right) U_0'(z)dz\\ &= -v\gamma\int_0^{\overline{T}} U_0'(z)^2dz + \int_0^{\overline{T}}\sum_{i=0}^{v-1} h(\omega\overline{T}i + \omega z + \alpha)U_0'(z)dz.\end{aligned}$$

In what follows, we study for simplicity the case with $v = 1$, so then (H2) becomes

(H3) $\quad \omega = \frac{2\pi u}{\overline{T}}$ for $u \in \mathbb{N}$.

Hence

$$\begin{aligned}M^u(\alpha) &= -\gamma\int_0^{\overline{T}} U_0'(z)^2dz + \int_0^{\overline{T}} h\left(\frac{2\pi u}{\overline{T}}z + \alpha\right) U_0'(z)dz\\ &= -\gamma\int_0^{\overline{T}} U_0'(z)^2dz - \frac{2\pi u}{\overline{T}}\int_0^{\overline{T}} h'\left(\frac{2\pi u}{\overline{T}}z + \alpha\right) U_0(z)dz.\end{aligned}$$

Example 1. To illustrate the theory, we consider $\varphi(U) = U + U^3$ and then equation from (H1) is the Duffing equation

$$U''(z) + U(z) + U^3(z) = 0 \tag{28}$$

possessing a family of periodic solutions

$$U_{0,a}(z) = a\,\mathrm{cn}\left(\sqrt{1+a^2}z\right)$$

for $a > 0$ with periods $\overline{T} = \overline{T}(a) = \frac{4K(k)}{\sqrt{1+a^2}}$, $k = \frac{a}{\sqrt{2+2a^2}}$. Note $U_{0,a}(0) = a$ and $U_{0,a}'(0) = 0$. Here cn is the Jacobi elliptic function, $K(k)$ is the complete elliptic function of the first kind and k is the elliptic modulus [16]. Moreover, we have

$$\overline{T}'(a) = \frac{8(E(k) - K(k)) - 4a^2K(k)}{a\sqrt{1+a^2}\,(2+a^2)} < 0,$$

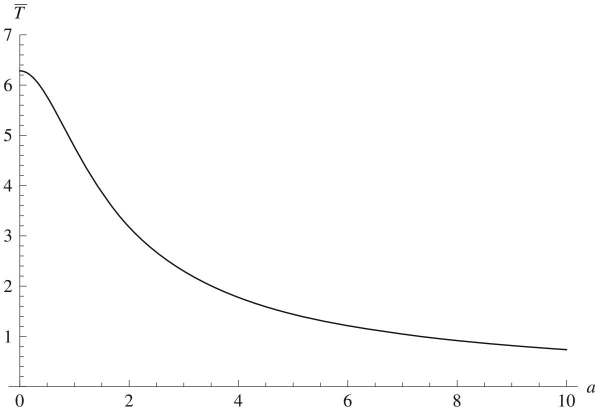

Fig. 1 The graph of function $\overline{T}(a)$

since $E(k) \leq K(k)$, where $E(k)$ is the complete elliptic function of the second kind. So $\overline{T}(a)$ is decreasing from $\overline{T}(0) = 2\pi$ to 0 (see Fig. 1). Then condition (H3) holds for any $\omega > 1$.

Now, we compute

$$\int_0^{\overline{T}(a)} U_{0,a}'(z)^2 dz = a^2\left(1+a^2\right)\int_0^{\overline{T}(a)} \mathrm{dn}^2\left(\sqrt{1+a^2}z\right)\mathrm{sn}^2\left(\sqrt{1+a^2}z\right)dz$$
$$= a^2\sqrt{1+a^2}\int_0^{4K(k)} \mathrm{dn}^2 z\,\mathrm{sn}^2 z dz = 2\frac{a^2}{k^2}\sqrt{1+a^2}\int_0^{2K(k)} \mathrm{dn}^2 z(1-\mathrm{dn}^2 z)dz$$
$$= 2(1+a^2)^{3/2}\int_0^{2K(k)}\left(\mathrm{dn}^2 z - \mathrm{dn}^4 z\right)dz = \frac{2}{3}\sqrt{1+a^2}\left((2+a^2)K(k)-2E(k)\right)$$

by using (2.1.10), (2.2.11–13), (2.5.3), Sect. 3.5 and (3.10.14) of [16]. Here dn and sn are the other Jacobi elliptic functions.

Now, we take $h(z) = \cos z$ and evaluate

$$\frac{2\pi u}{\overline{T}(a)}\int_0^{\overline{T}(a)} h'\left(\frac{2\pi u}{\overline{T}(a)}z+\alpha\right)U_{0,a}(z)dz$$
$$= -\frac{2\pi a u}{\overline{T}(a)}\int_0^{\overline{T}(a)} \sin\left(\frac{2\pi u}{\overline{T}(a)}z+\alpha\right)\mathrm{cn}\left(\sqrt{1+a^2}z\right)dz$$
$$= -\frac{\pi a u}{2K(k)}\int_0^{4K(k)} \sin\left(\frac{\pi u}{2K(k)}z+\alpha\right)\mathrm{cn}\; z\,dz$$
$$= -\frac{\pi a u}{2K(k)}\left(\sin\alpha\int_{-2K(k)}^{2K(k)}\cos\left(\frac{\pi u}{2K(k)}z\right)\mathrm{cn}\; z\,dz + \cos\alpha\int_{-2K(k)}^{2K(k)}\sin\left(\frac{\pi u}{2K(k)}z\right)\mathrm{cn}\; z\,dz\right)$$
$$= -\frac{\pi a u}{2K(k)}\sin\alpha\int_0^{4K(k)}\cos\left(\frac{\pi u}{2K(k)}z\right)\mathrm{cn}\; z\,dz$$
$$= \frac{(-1)^u-1}{2}\,\frac{\pi^2 a u}{kK(k)}\sin\alpha\,\mathrm{sech}\left(\frac{\pi K(\sqrt{1-k^2})u}{2K(k)}\right)$$

since $\cos z$ and $\mathrm{cn}\, z$ are even and $\sin z$ is odd, and the Fourier expansion of cn (see (8.7.7) and (2.6.5) of [16]) are used as well. Summarizing, the Melnikov function is now

$$\begin{aligned} M^u(\alpha) &= -\gamma\frac{2}{3}\sqrt{1+a^2}\left((2+a^2)K(k)-2E(k)\right) \\ -\frac{(-1)^u-1}{2}\frac{\pi^2 au}{kK(k)}\sin\alpha\,\mathrm{sech}&\left(\frac{\pi K(\sqrt{1-k^2})u}{2K(k)}\right), \quad k=\frac{a}{\sqrt{2+2a^2}}. \end{aligned} \tag{29}$$

Clearly, we need $u \in \mathbb{N}$ to be odd. Then to have a simple zero of $M^u(\alpha)$, we need

$$\gamma\frac{\sqrt{2}K(k)\left((2+a^2)K(k)-2E(k)\right)}{3\pi^2 u}\cosh\left(\frac{\pi K(\sqrt{1-k^2})u}{2K(k)}\right)<1, \quad k=\frac{a}{\sqrt{2+2a^2}}.$$

Setting

$$\Lambda(a,u) := \frac{\sqrt{2}K(k)\left((2+a^2)K(k)-2E(k)\right)}{3\pi^2 u}\cosh\left(\frac{\pi K(\sqrt{1-k^2})u}{2K(k)}\right)<1,$$

$k=\frac{a}{\sqrt{2+2a^2}}$, we see that

$$\gamma < \frac{1}{\Lambda(a,u)} \tag{30}$$

gives the magnitude for the damping in order to apply Theorem 3. Note $\Lambda(a,u) > 0$ for any $a > 0$, $u > 0$, $\Lambda(0,u) = 0$ and $\Lambda(a,u) \to \infty$ as $a \to \infty$.

4 Bifurcation Results for Asymptotic Travelling Waves

In this section, we first consider, instead of (H1), the following one

(C1) $\varphi(0) = 0$, $\varphi'(0) < 0$ and $U''(z) + \varphi(U(z)) = 0$ has an asymptotic solution $\Gamma \in C^2(\mathbb{R},\mathbb{R})$ such that $\lim_{|z|\to\infty}\Gamma(z) = 0$ and $\lim_{|z|\to\infty}\Gamma'(z) = 0$.

Remark 4. We may again suppose like in Remark 3 that Γ is even.

Then

$$\Gamma_\omega(z) := \Gamma(z/\omega)$$

satisfies $\omega^2\Gamma_\omega''(z) + \varphi(\Gamma_\omega(z)) = 0$, $\lim_{|z|\to\infty}\Gamma_\omega(z) = 0$ and Γ_ω is even. Next, when we take $U_\omega = 0$ in Sect. 3, then $X_1 = \{0\}$ and $Y_1 = \{0\}$. So there is no bifurcation Eq. (25). This implies the following result.

Theorem 4. *Suppose (C1). Then there is a* $\delta > 0$ *such that for any* $\varepsilon \in (-\delta, \delta)$ *there is a unique* 2π*-periodic solution* $W_\varepsilon(z)$ *of (21) with* $W_\varepsilon(z) = O(\varepsilon)$.

Remark 5. A proof of Theorem 4 follows also directly from Sect. 2. Under (C1), the equation $\omega^2 U''(z) + \varphi(U(z)) = 0$ has a hyperbolic/nonresonant equilibrium $U_\omega(z) = 0$. The linearization at U_ω is $\omega^2 U''(z) + \varphi'(0)U(z) = 0$, which has no nonzero periodic solutions. So for any periodic perturbation, i.e. also for (21), there is a small periodic solution. This is a statement of Theorem 4.

Now, we follow the standard homoclinic Melnikov method [3, 12] to (21) based on the Lyapunov-Schmidt method, but for reader's convenience we again present some details. First, we take Banach spaces

$$X := \left\{U \in C^2(\mathbb{R}, \mathbb{R}) \mid \|U\|_{2,\infty} = \|U\|_\infty + \|U'\|_\infty + \|U''\|_\infty < \infty\right\},$$
$$Y := \{U \in C(\mathbb{R}, \mathbb{R}) \mid \|U\|_\infty < \infty\}$$

with the usual supremum norm $\|U\|_\infty = \sup_{z\in\mathbb{R}} |U(z)|$. Then we split

$$Y = Y_1 \oplus Y_2, \quad X = X_1 \oplus X_2$$

with

$$Y_1 := \text{span}[\Gamma_\omega'], \quad Y_2 := \left\{U \in Y \mid \int_{-\infty}^{\infty} U(z)\Gamma_\omega'(z)dz = 0\right\},$$
$$X_1 := Y_1, \quad X_2 := \left\{U \in Y \mid \int_{-\infty}^{\infty} U(z)\Gamma_\omega'(z)dz = 0\right\}.$$

Next, we take the projections $P : Y \to Y_1$ and $Q = \mathbb{I} - P : Y \to Y_2$ defined as

$$PU := \frac{\int_{-\infty}^{\infty} U(s)\Gamma_\omega'(s)ds}{\int_{-\infty}^{\infty} \Gamma_\omega'^2(s)ds} \times \Gamma_\omega.$$

Now, we use the following well-known analogy of Lemma 3 [3, 12].

Lemma 4. *Equation* $\omega^2 V''(z) + \varphi'(U_\omega(z))V(z) = \tilde{Y}(z) \in Y$ *has a solution* $V \in X$ *if and only if* $\tilde{Y} \in Y_2$. *This solution is unique requiring* $V \in X_2$. *Moreover, there is a constant* $c_\omega > 0$ *such that* $\|V\|_{2,\infty} \le c_\omega \|\tilde{Y}\|_\infty$.

Consequently, we can follow the approach of Sect. 3 as in [3, 12] to derive the following result.

Theorem 5. *Suppose (C1). If there is a simple zero* β_0 *of the Melnikov function*

$$M(\beta) := \int_{-\infty}^{\infty} \left(-\gamma\Gamma'(z) + h(\omega z + \beta)\right) \Gamma'(z)dz. \tag{31}$$

Then there is a $\theta > 0$ *such that for any* $0 \neq \varepsilon \in (-\theta, \theta)$ *there is a unique bounded solution* $U(z)$ *of (21) on* $\mathbb{R}$ *with*

$$U(z) = \Gamma\left(\frac{z-\beta_0}{\omega}\right) + O(\varepsilon).$$

Next, putting $U(z) = W_\varepsilon(z) + W(z)$ in (21), where $W_\varepsilon(z)$ is a small 2π-periodic solution from Theorem 4, we get

$$\begin{aligned} \omega^2 W''(z) + \varphi(W_\varepsilon(z) + W(z)) - \varphi(W_\varepsilon(z)) \\ -\varepsilon\lambda\omega^2\left(W''(z-p) + W''(z+p)\right) + \varepsilon\gamma\omega W'(z) = 0. \end{aligned} \tag{32}$$

Now, we take Banach spaces

$$\begin{aligned} X_\varsigma &:= \left\{U \in C^2(\mathbb{R}, \mathbb{R}) \mid \|U\|_{2,\varsigma} = \|U\|_\varsigma + \|U'\|_\varsigma + \|U''\|_\varsigma < \infty\right\}, \\ Y_\varsigma &:= \left\{U \in C(\mathbb{R}, \mathbb{R}) \mid \|U\|_\varsigma < \infty\right\} \end{aligned}$$

with the weighted supremum norm $\|U\|_\varsigma = \sup_{z\in\mathbb{R}} |U(z)|\, \mathrm{e}^{\varsigma|z|}$ for a sufficiently small $\varsigma > 0$. Then we can repeat the above arguments to solve (32) in X_ς under assumptions of Theorem 5. Indeed, Lemma 4 holds in X_ς and Y_ς [11]. Denoting $\Upsilon_\omega := \frac{d}{d\varepsilon} W_0$, (32) has the form

$$\begin{aligned} 0 &= \omega^2 W''(z) + \varphi(W(z)) + \varphi(W_\varepsilon(z) + W(z)) - \varphi(W_\varepsilon(z)) - \varphi(W(z)) \\ &\quad -\varepsilon\lambda\omega^2\left(W''(z-p) + W''(z+p)\right) + \varepsilon\gamma\omega W'(z) \\ &= \omega^2 W''(z) + \varphi(W(z)) + \varepsilon\left(\varphi'(W(z)) - \varphi'(0)\right)\Upsilon_\omega(z) + O(\varepsilon^2) \\ &\quad -\varepsilon\lambda\omega^2\left(W''(z-p) + W''(z+p)\right) + \varepsilon\gamma\omega W'(z). \end{aligned} \tag{33}$$

Note $U_\omega \in X_\varsigma$ and $\omega^2\Upsilon_\omega''(z) + \varphi'(0)\Upsilon_\omega(z) - h(z) = 0$. Equation (33) has a similar form as (21), so we set $W(z+\beta) = U_\omega(z) + W_1(z)$ as in (22) and apply the above approach to get the Melnikov function

$$\begin{aligned} M(\beta) &:= \int_{-\infty}^{\infty} \Big(\lambda\omega^2\left(U_\omega''(z-p) + U_\omega''(z+p)\right) - \gamma\omega U_\omega'(z) \\ &\qquad - \left(\varphi'(U_\omega(z)) - \varphi'(0)\right)\Upsilon_\omega(z+\beta)\Big)U_\omega'(z)dz \\ &= \int_{-\infty}^{\infty} \left(-\gamma\omega U_\omega'(z) - \varphi'(U_\omega(z))\Upsilon_\omega(z+\beta) - \omega^2\Upsilon_\omega''(z+\beta) + h(z+\beta)\right) U_\omega'(z)dz, \end{aligned} \tag{34}$$

since U_ω is even. By using the integration by parts, we derive

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\varphi'(U_\omega(z))\Upsilon_\omega(z+\beta) + \omega^2\Upsilon_\omega''(z+\beta)\right) U_\omega'(z)dz \\ &= \int_{-\infty}^{\infty} \left(\frac{d}{dz}\left[\varphi(U_\omega(z))\right]\Upsilon_\omega(z+\beta) + \omega^2\Upsilon_\omega''(z+\beta)U_\omega'(z)\right) dz \\ &= \int_{-\infty}^{\infty} \left(-\varphi(U_\omega(z))\Upsilon_\omega'(z+\beta) - \omega^2\Upsilon_\omega'(z+\beta)U_\omega''(z)\right) dz = 0. \end{aligned}$$

Hence (34) has the form

$$\begin{aligned} M(\beta) &= \int_{-\infty}^{\infty} \left(-\gamma\omega U_\omega'(z) + h(z+\beta)\right) U_\omega'(z)dz \\ &= \frac{1}{\omega}\int_{-\infty}^{\infty} \left(-\gamma\Gamma'(z/\omega) + h(z+\alpha)\right)\Gamma'(z/\omega)dz \\ &= \int_{-\infty}^{\infty} \left(-\gamma\Gamma'(z) + h(\omega z+\alpha)\right)\Gamma'(z)dz, \end{aligned} \tag{35}$$

which is just (31). Consequently, under the assumptions of Theorem 5, there is a $\theta_1 > 0$ such that for any $0 \neq \varepsilon \in (-\theta_1, \theta_1)$ there is a unique solution $W(z) \in Y_\varsigma$ of (32) with

$$W(z) = \Gamma\left(\frac{z-\beta_0}{\omega}\right) + O(\varepsilon)$$

in Y_ς. Then $U_1(z) := W_\varepsilon(z) + W(z) = \Gamma\left(\frac{z-\beta_0}{\omega}\right) + O(\varepsilon)$ is a bounded solution of (21) on $\mathbb{R}$ with the same property as $U(z)$ in Theorem 5. The uniqueness of this solution gives $U(z) = U_1(z)$. Consequently, the solution predicted in Theorem 5 exponentially tends to the small periodic solution.

Example 2. To illustrate the theory, we consider $\varphi(U) = -U + U^3$ and then equation from (C1) is the Duffing equation

$$U''(z) - U(z) + U^3(z) = 0 \tag{36}$$

possessing a homoclinic solution

$$\Gamma(z) = \sqrt{2}\,\text{sech}\, z.$$

Again $h(z) = \cos z$. Then the Melnikov function (31) is now [12, p. 191]

$$\begin{aligned} M(\beta) &:= -\sqrt{2}\int_{-\infty}^{\infty}\left(\sqrt{2}\gamma\,\text{sech}\, z\tanh z + \cos(\omega z+\beta)\right)\text{sech}\, z\tanh z dz \\ &= -\frac{4\gamma}{3} + \sqrt{2}\omega\pi\,\text{sech}\left(\frac{\omega\pi}{2}\right)\sin\beta. \end{aligned}$$

If

$$\gamma < \frac{3\sqrt{2}}{4}\omega\pi\,\text{sech}\left(\frac{\omega\pi}{2}\right), \tag{37}$$

then $M(\beta)$ has a simple zero and so Theorem 5 can be applied.

Next, we consider the following condition.

(C2) $\varphi(\pm 1) = 0$, $\varphi'(\pm 1) < 0$ and $U''(z) + \varphi(U(z)) = 0$ has an asymptotic solution $\Gamma \in C^2(\mathbb{R}, \mathbb{R})$ such that $\lim_{z \to \pm\infty} \Gamma(z) = \pm 1$ and $\lim_{|z| \to \infty} \Gamma'(z) = 0$.

Remark 6. Now $U'(z) > 0$ on $\mathbb{R}$. Next $\Gamma_-(z) := \Gamma(-z)$ is a solution satisfying $\lim_{z \to \pm\infty} \Gamma_-(z) = \mp 1$ and $\lim_{|z| \to \infty} \Gamma_-'(z) = 0$. So Γ and Γ_- create a heteroclinic cycle. Next, we can suppose that $\Gamma(0) = 0$. If φ is odd then Γ is odd. So Γ' is even and Γ'' is odd.

Similarly, we get 2π-periodic solutions $W_{\varepsilon,\pm}(z) = \pm 1 + O(\varepsilon)$ of (21), for any $\varepsilon \neq 0$ small. Now, we can repeat the above approach to derive the Melnikov function

$$M(\beta) := \int_{-\infty}^{\infty} \left(\lambda \left(\Gamma'' \left(z - \frac{p}{\omega}\right) + \Gamma'' \left(z + \frac{p}{\omega}\right)\right) - \gamma \Gamma'(z) + h(\omega z + \beta)\right) \Gamma'(z) dz. \tag{38}$$

Remark 7. By Remark 6 we know that if φ is odd then Γ' is even and Γ'' is odd. Then (38) possesses the form of (31).

Under assumptions of Theorem 5 with (38), we get a bounded solution $U(z)$ of (21) with the same properties and exponentially tending to the small periodic solutions $W_{\varepsilon,\pm}$. We note that to show these exponential attractions, we consider (32) for $W_{\varepsilon,\pm}$ on the Banach spaces

$$X_\varsigma^\pm := \left\{U \in C^2(\mathbb{R}_\pm, \mathbb{R}) \mid \|U\|_{2,\varsigma} = \|U\|_\varsigma + \|U'\|_\varsigma + \|U''\|_\varsigma < \infty\right\},$$
$$Y_\varsigma^\pm := \left\{U \in C(\mathbb{R}_\pm, \mathbb{R}) \mid \|U\|_\varsigma < \infty\right\},$$

respectively.

Example 3. To illustrate the theory, we consider $\varphi(U) = U - U^3$ and then equation from (C2) is the Duffing equation

$$U''(z) + U(z) - U^3(z) = 0 \tag{39}$$

possessing a heteroclinic solution

$$\Gamma(z) = \tanh(z/\sqrt{2}).$$

Again $h(z) = \cos z$. Clearly φ is odd, so by Remark 7, the Melnikov function (38) has the form (31), so it is given by [3]

$$\begin{aligned} M(\beta) &:= \frac{1}{2} \int_{-\infty}^{\infty} \left(-\gamma \operatorname{sech}^2\left(\frac{z}{\sqrt{2}}\right) + \sqrt{2} \cos(\omega z + \beta)\right) \operatorname{sech}^2\left(\frac{z}{\sqrt{2}}\right) dz \\ &= \frac{\sqrt{2}}{3} \left(-2\gamma + 3\omega\pi \cos\beta \operatorname{csch}\left(\frac{\omega\pi}{\sqrt{2}}\right)\right). \end{aligned}$$

If

$$\gamma < \frac{3}{2}\omega\pi \operatorname{csch}\left(\frac{\omega\pi}{\sqrt{2}}\right),$$

then $M(\beta)$ has a simple zero and so Theorem 5 can be applied.

Remark 8. Under either (C1) or (C2) there is an accumulation of periodic travelling waves of (21) on the asymptotic travelling waves predicted in Theorem 5 with periods tending to infinity. This follows from similar arguments as in [12, p. 197]. So we have the so called blue sky catastrophe [6].

Remark 9. By following [5, 6], method of Sects. 3 and 4 can be applied to (19) to show the bifurcation of periodic and asymptotic breathers. Related results are given in [4, 7–10].

Finally, we construct a non-odd φ satisfying (C2). We look for φ in the form $\varphi(U) = U(1-U^2)\left(a_1U^3 + a_2U^2 + a_3U + 1\right)$ for some $a_i \in \mathbb{R}$, $i = 1, 2, 3$. Note $\varphi'(0) = 1$. Then

$$\Phi(U) := \int \varphi(U)\, dU = \frac{U^2}{2} + \frac{a_3U^3}{3} - \frac{U^4}{4} + \frac{a_2U^4}{4} + \frac{a_1U^5}{5} - \frac{a_3U^5}{5} - \frac{a_2U^6}{6} - \frac{a_1U^7}{7}.$$

The condition $\Phi(-1) = \Phi(1)$ implies $3a_1 + 7a_3 = 0$, so we take $a_1 = -7a$, $a_3 = 3a$ and $a_2 = b$. Then

$$\varphi(U) = U(1-U^2)\left(1 + bU^2 + aU\left(3 - 7U^2\right)\right) \tag{40}$$

and

$$\Phi(U) = \frac{1}{12}U^2\left(6 + 3(-1+b)U^2 - 2bU^4 + 12aU\left(-1+U^2\right)^2\right).$$

Set

$$\mathcal{M} = \left\{(a,b) \in \mathbb{R}^2 \mid \min_{U\in[-1,1]} \left(1 + bU^2 + aU\left(3 - 7U^2\right)\right) > 0\right\}.$$

Clearly, if $(a, b) \in \mathcal{M}$ then φ given by (40) satisfies (C2). Next, since $\max_{U\in[0,1]} |U\left(3 - 7U^2\right)| = 4$. Then for any $b \geq 0$ and $|a| < 1/4$ it holds

$$1 + bU^2 + aU\left(3 - 7U^2\right) \geq 1 - 4|a| > 0.$$

Hence $\mathcal{M}$ is nonempty. Of course, further computations must be done numerically.

5 Numerical Results

To illustrate the theoretical results obtained in the previous sections, we have solved the governing equation (4) (cf. (21)) numerically. The advance-delay equation is solved using a pseudo-spectral method. We express the solution U in a Fourier series

$$U(z) = \sum_{j=1}^{J} \left[A_j \cos\left((j-1)\tilde{k}z\right) + B_j \sin\left(j\tilde{k}z\right)\right], \tag{41}$$

where $\tilde{k} = 2\pi/L$ and $-L/2 < z < L/2$. The Fourier coefficients A_j and B_j are then found by requiring the series to satisfy (4) at several collocation points. Hence, $2J$ collocation points are required, which are chosen with uniform grid points. The stability of a solution obtained from (4) is then determined numerically by evolving it through the original equation (1). The governing equation is integrated using a Runge-Kutta method of order four with periodic boundary conditions. In the following, we only illustrate the results of Sect. 4 on the existence of asymptotic travelling waves. In particular, we consider the nonlinearity discussed in Example 2. The nonlinearity studied in Example 3 will be mentioned briefly. We have considered periodic wave solutions as well, but they are not presented here as calculations can be done rather straightforwardly using the numerical method.

It is important to note that the physically relevant range for the coupling parameter λ is $|\lambda| < 1/2$ for the following reason. When we consider (1) in $\ell^2 \times \ell^2$, then we need to solve the equation

$$\lambda x_{n-1} - x_n + \lambda x_{n-1} = y_n \quad n \in \mathbb{Z}$$

in ℓ^2, i.e. the linear equation $D(\lambda)x = y$, $x, y \in \ell^2$. The spectrum of the corresponding left-hand side linear self-adjoint operator $D(\lambda)$ is $\sigma(D(\lambda)) = [-2|\lambda| - 1, 2|\lambda| - 1]$ (see [25, p. 20]). When $|\lambda| < 1/2$, then $\operatorname{dist}\{0, \sigma(D(\lambda))\} \geq 1 - 2|\lambda| > 0$. This implies that $\|D(\lambda)^{-1}\| \leq \frac{1}{1-2|\lambda|}$. On the other hand, if $|\lambda| \geq 1/2$ then $0 \in \sigma(D(\lambda)) = 0$, and (1) is problematic. The situation is different on $\ell^\infty \times \ell^\infty$, since we can consider periodic boundary conditions $i_j = i_{n+j}$, $q_j = q_{n+j}$, $j \in \mathbb{Z}$. So we truncate D and solve a linear equation

$$\lambda x_2 - x_1 + \lambda x_n = y_1\,,$$

$$\vdots$$

$$\lambda x_1 - x_n + \lambda x_{n-1} = y_n\,.$$

The spectrum of the corresponding left-hand side linear symmetric matrix $D_n(\lambda)$ is $\sigma(D_n(\lambda)) = \{2\lambda \cos \frac{2\pi}{n} j - 1 \mid j = 0, \cdots, n-1\}$. So the above linear equation is $D_n(\lambda)x = y$. When $|\lambda| < 1/2$, then $\operatorname{dist}\{0, \sigma(D_n(\lambda))\} \geq 1 - 2|\lambda| > 0$ and

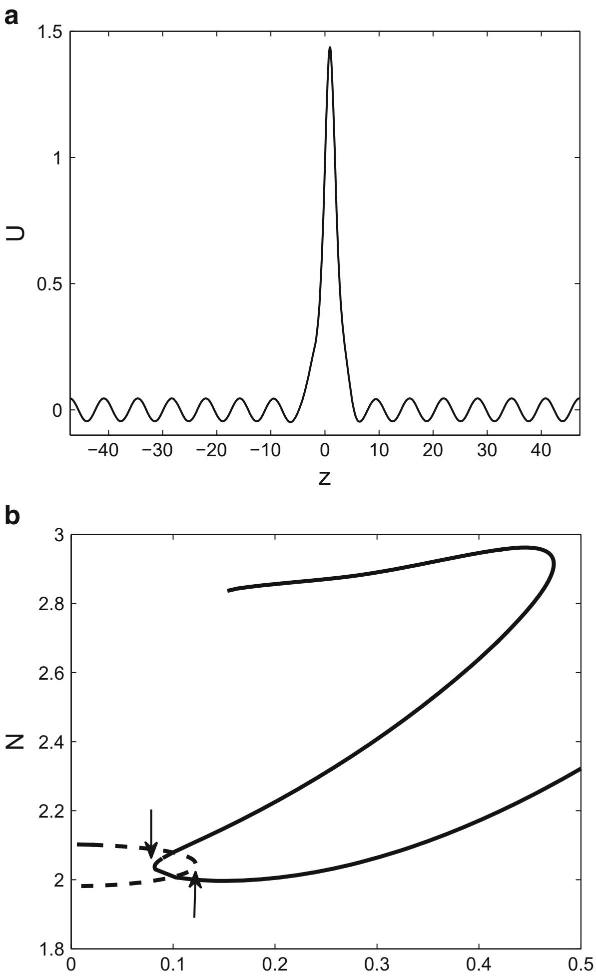

Fig. 2 (**a**) An asymptotic travelling wave for a Duffing nonlinearity $\varphi = -U + U^3$. (**b**) Continuations of the solution in (**a**) for varying γ (*dashed*) and f (*solid*). On the vertical axis is the solution norm (see the text)

$\|D_n(\lambda)^{-1}\| \leq \frac{1}{1-2|\lambda|}$ independently of n. Moreover, $\sigma(D_n(\lambda)) \subset (-\infty, 0)$. On the other hand, if $|\lambda| > 1/2$ then $\sigma(D_n(\lambda)) \cap (0, \infty) \neq \emptyset$. Therefore, in this case one will observe a blow-up at finite time. Nevertheless, $D_n(\lambda)$ could still be invertible, which explains why periodic travelling waves could also exist for $|\lambda| > 1/2$ (see Remark 1).

For the nonlinearity discussed in Example 2, i.e. $\varphi = -U + U^3$, shown in Fig. 2a is the profile of an asymptotic wave for $\lambda = \gamma = f = 0.1$, $\omega = 1$ and $p = \pi$. The numerical solution is computed using $L = 30\pi$ and $J = 100$. We have used larger values of J as well, but we did not see any significant quantitative difference. One can observe that the single-hump profile in Fig. 2a is accompanied by periodic waves as suggested by Theorem 4. The asymptotic wave is also found in the existence region given by Theorem 5 (see also Eqs. (35) and (37)).

Regarding the existence of the asymptotic wave, we have performed numerical continuations of the wave in the panel by varying one parameter using a pseudo-arclength method. Shown in Fig. 2b is the continuation of the figure as the damping parameter γ or the driving amplitude f varies that is shown in dashed and solid line, respectively. The vertical axis is the norm of the solution

$$N = \sqrt{\int_{-L/2}^{L/2} |U|^2 \, dx}.$$

Note that there are two intersections between the dashed and the solid lines. The wave in Fig. 2a corresponds to the upper intersection.

For fixing $f = 0.1$ and varying γ, we obtained numerically that there is a saddle-node bifurcation at $\gamma \approx 0.122$. In Fig. 2b, the bifurcation is indicated by the right arrow. From the Melnikov function (37), the approximate boundary is calculated as $\gamma \approx 0.133$, which is quantitatively close to the numerical value. As for varying the driving amplitude f while fixing $\gamma = 0.1$, we also observed saddle-node bifurcations. Upon decreasing f, there is a saddle-node bifurcation between the middle branch, corresponding to single-hump asymptotic waves, and the lower branch that corresponds to periodic waves. The bifurcation, which is indicated by the left arrow, occurs at $f \approx 0.082$. From Theorem 5 (cf. Eq. (37)), the existence for asymptotic waves is bounded by $f \approx 0.075$, which agrees quite well with the numeric.

From the solution in Fig. 2a, when we increase the driving amplitude f, we also observe another saddle-node bifurcation. The upper branch in the panel corresponds to asymptotic waves with double-hump. We did not study the multi-hump waves further because it is beyond the scope of the present paper.

After analysing the existence, next we consider the stability of the asymptotic waves. For the profile shown in Fig. 2a, depicted in Fig. 3 is the typical time-dynamics of the wave obtained from integrating the governing equation (1). One can observe that the travelling wave is strongly unstable. The hump could only travel for one site before the background becomes excited and destroys the localised profile. The instability is naturally expected due to the fact that the zero solution (when there is no drive) forming the background of the asymptotic wave is unstable, i.e. it is a saddle point. For that reason, we believe that all the branches in Fig. 2b correspond to unstable solutions. For the same reason, the asymptotic waves in Example 3 would also be unstable as the background, i.e. $U = 1$ when there is no drive, is also a saddle point. Moreover, differently from the zero background in Example 2, the instability in this case will create an unbounded blow-up (see Ref. [1] for a related problem).

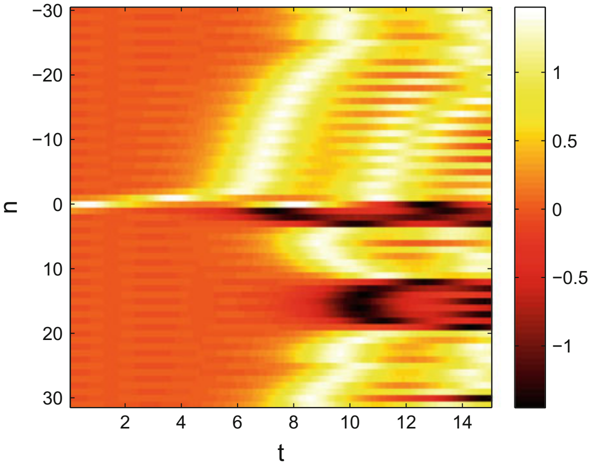

Fig. 3 Time dynamics of the wave shown in Fig. 2a. Plotted is the top view of $q_n(t)$

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