On the Solution of Fluid Surface Flow Caused by a Line Source Using the Decomposition Method

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Abstract

An approximate model of fluid flow caused by a line source is considered. An analytical solution from this approximate model is got by the decomposition method. It is found that the accuracy of this new method can be very much improved by manipulating the zeroth component of the decomposition series.

Keywords

Fluid surface flow, Froude number, The Adomian decomposition method

1 Introduction

This paper is concerned with the Adomian decomposition method. This method in the recent years has been widely considered. The decomposition method is considered as a generalization of Taylor series, but the expansion is about a function instead of a point. The outline of the method is presented in section 3. This method is then applied to the model of the steady 2-D fluid flow as shown in Figure 1. This application is presented in section 4.

The domain of the flow in the model is bounded by a vertical and horizontal walls which are perpendicular at the corner O. The fluid is assumed to be incompressible and inviscid. The line source with a strength m/π is placed along the vertical wall at the distance S from O. At the surface right above the source, there is a stagnation point A where the magnitude of the velocity at this point is zero. Far downstream, the flow tends to uniform with the speed U and the fluid depth H.

Some authors have looked for the solution of the flow at the surface caused by this line source for small Froude number. Mekias & Vanden-Broeck [3] derive the model using the image method. Using the rigid lid approximation, Wiryanto [5] also finds an approximate model which has a good agreement with the result got by Mekias & Vanden-Broeck. In section 2 we present the derivation of this rigid lid approximation briefly.

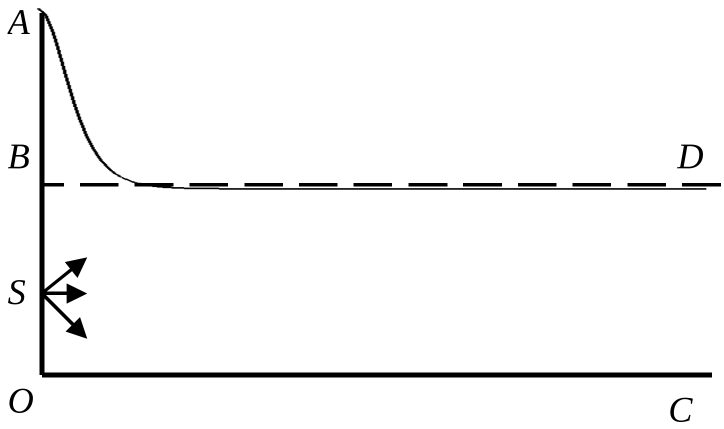


Figure 1: Flow domain in the z-plane.

Our objective of applying this decomposition method is to gain an analytic solution in a rapidly convergent series. In section 4 we see that a modification to the zeroth order of the Adomian series can increase the convergence of the method.

2 Derivation of the rigid lid model

At the free surface with constant pressure, we have Bernoulli's equation

$$\frac{1}{2}(u^2+v^2)+gy=\text{constant}, \tag{1}$$

where u and v are the speed in a horizontal and vertical directions respectively and g is the gravity acceleration. From the assumption given in section 1, the righthand side of (1) is $1/2U^2+gH$. In nondimensionalising the variables, we use H for the length scale and $U=m/H$ for the velocity scale. Hence, (1) becomes

$$\frac{1}{2}F^2(u^2+v^2)+y=\frac{1}{2}F^2+1, \tag{2}$$

with $F=m/\sqrt{(}gH^3)$ is Froude number and the strength of the source reduces to $1/\pi$.

Mathematically, then the problem is to determine the magnitude of the velocity u^2+v^2 satisfying Laplace's equation within the domain and conditions of no flux through the boundaries. By defining the complex potential $f=f(z)$ which gives $u-iv=f_z$, except at the source, we only know

$$f(z)\to\tfrac{m}{\pi}\ln(z-iS)\quad\text{for}\quad z\to iS.$$

From (2) we know that if $F=0$ the free surface reduces to the line $y=1$. We will use the velocity of the flat free boundary which is usually called rigid lid to approximate the velocity at the free surface of small Froude number flow.

To get the velocity for the flat boundary, we will use Schwarz-Christoffel transformation (see [4]). Using this conformal mapping which relates the complex variable

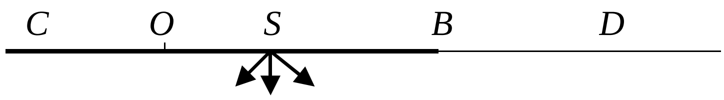


Figure 2: Flow domain in the ζ-plane.

$z = x + iy$ and the artificial variable $\zeta = \xi + i\eta$, we will map the two corners O and B to $\zeta = -1$ and $\zeta = 1$ respectively. The source at $z = iS$ corresponds to $\zeta = b$. The sketch of the flow domain in ζ-plane is shown in Figure 2.

At this artificial flow domain, we know that the complex potential is given by

$$f = \frac{1}{\pi} \ln(\zeta - b). \tag{3}$$

The Schwarz-Christoffel transformation is given by

$$\frac{dz}{d\zeta} = \frac{K}{\sqrt{\zeta^2 - 1}}$$

where K is a constant determined using the corresponding points O and B. These relations give an explicit expression

$$\zeta = \cosh \pi(z - i).$$

Substituting the above relation to (3) and then differentiating with respect to z, we obtain

$$u^2 + v^2 = -\frac{(\cos \pi y)^2 - (\cosh \pi x)^2}{-2\cosh \pi x \cos \pi y \cos \pi S + (\cosh \pi x)^2 - (\sin \pi y)^2 + (\cos \pi S)^2}. \tag{4}$$

Wiryanto [5] then evaluates the above magnitude of the velocity to approximate the elevation of the surface, i.e.

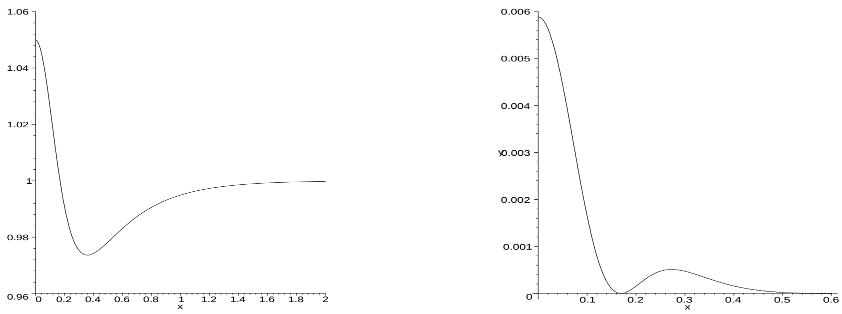
$$y \approx 1 + \frac{1}{2} F^2 \left(1 - (u^2 + v^2)\right)\Big|_{y=1}, \tag{5}$$

where the equation is then of $\mathcal{O}(F^2)$. Plot of the solution is given in Figure 3(a).

In this consideration, we suggest to use the velocity of the flat boundary to approximate the elevation of the free surface without evaluating at $y = 1$, e.g. we substitute $u^2 + v^2$-term in equation (2) with equation (4). Hence, we have an implicit equation of y. Plot of the difference of the solution given by (5) and the solution given by the implicit equation is presented in Figure 3(b).

We see that the difference of these two equations is relatively so significance. The difference is determined by the value of the Froude number and the distance of the source from the bottom. Further analysis is needed to know which one is better to approximate the real solution. But we get a feeling that the first equation is approximating the function from 'above' and the second one is approximating from 'below'.

The present paper is dealing with looking for the analytical solution of the implicit function using the decomposition method.



(a) (b)

Figure 3: Plot of (a) equation (5) for $S = 0.7$ and $F = \sqrt{0.1}$ (b) the difference of the solution given by the explicit equation (5) and the implicit one (2).

3 Outline of the Adomian decomposition method

Mainly, the Adomian decomposition method [1, 2] is a method which uses decomposition of the unknown solution $y(x)$ into a sum of components defined by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{6}$$

and expressing the nonlinear term in the Adomian's polynomials. It is justified by [6] that this method is very reliable and effective. This method provides the solution in terms of rapid convergent series.

By defining $j = \sum_{n=0}^{\infty} \lambda^n y_n$, a nonlinear term $N(y)$ can be written in Adomian's polynomials

$$N(y) = \sum_{n=0}^{\infty} A_n$$

with the components are defined as

$$\begin{aligned} A_0 &= N(y_0) \\ A_1 &= \left(\frac{\partial N(y_0)}{\partial y_0} \frac{\partial j}{\partial \lambda} \right)\bigg|_{\lambda=0} \\ A_2 &= \left(\frac{\partial^2 N(y_0)}{\partial y_0^2} \left(\frac{\partial j}{\partial \lambda} \right)^2 + \frac{\partial N(y_0)}{\partial y_0} \frac{\partial^2 j}{\partial \lambda^2} \right)\bigg|_{\lambda=0} \\ &\vdots \end{aligned} \tag{7}$$

An example of the program to count this Adomian's polynomials is given in appendix.

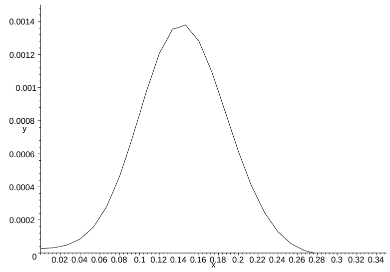


Figure 4: The difference of the Adomian series (9) with 8 terms and the real solution of the implicit equation (2).

4 Application of the method to the problem

To solve the implicit equation (2), substitute (6) and (7) to obtain

$$\sum_{n=0}^{\infty} y_n(x) \approx 1 + \frac{1}{2}F^2 \left(1 - \sum_{n=0}^{\infty} A_n(x)\right) \tag{8}$$

where we already wrote the nonlinear term $u^2 + v^2$ in the Adomian's polynomials. The components y_0, y_1, y_2, $\cdots$ of $y(x)$ in (6) are given by a recurrence relationship

$$\begin{aligned} y_0(x) &= 1 + \tfrac{1}{2}F^2, \\ y_{n+1}(x) &= -\tfrac{1}{2}F^2 A_n(x), n \geq 0. \end{aligned} \tag{9}$$

As a result of (9), the terms y_0, y_1, y_2, $\cdots$, are easily calculated. Plot of the difference between the series of 8-terms and the real solution is shown in Figure 4.

Inspired by (5), we are interested in seeing the effect of choosing the first two terms as

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= \tfrac{1}{2}F^2 \left(1 - A_0(x)\right), \\ y_{n+1}(x) &= -\tfrac{1}{2}F^2 A_n(x), n \geq 1. \end{aligned} \tag{10}$$

We then know that the difference of the real solution and this approximate function with 8-terms is of order 10^{-5}. Hence, we just found an example where the accuracy of the Adomian decomposition method might be very much improved by modifying the first some terms.

5 Conclusions

We have performed a model for fluid surface flow caused by a line source with small Froude number. We approximate the solution of the model by using rigid lid model. The Adomian decomposition method has been applied to get the analytical solution of the implicit equation given by the approximate model. It has been found also an example which shows that a modification of the first some terms can improve the speed of the convergence of the method.

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A Maple program to count the Adomian's polynomials

```
> restart:
> N:=j^2: # the nonlinear term
> ni:=7: # number of iterations
> c[0,0]:=1:
  c[1,0]:=1:
> for ik from 1 to ni do
   c[0,ik]:=0:
  od:
> A[0]:=subs(j=y[0],N):
> K:=lambda -> sum(y[k]*lambda^k,k=0..ni):
> for n from 1 to ni do
   for nu from 1 to n do
   c[nu,n]:=diff(c[nu,n-1],lambda)+c[nu-1,n-1]*diff(K(lambda),lambda);
   od;
   A[n]:=1/n!*subs(j=y[0],lambda=0,sum(c[v,n]*diff(uv2,j$v),v=1..n));
  od:
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References

[1] G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer Academic Publishers, Dordrecht, 1989.

[2] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, 1994.

[3] H. Mekias & J.M. Vanden-Broeck, Subcritical flow with a stagnation point due to a source beneath a free surface, *Phys. Fluids A* **3**:2652-2658, 1991.

[4] M. Waker, *The Schwarz-Christoffel Transformation and its Applications: A Simple Exposition*, Dover Publications, New York, 1964.

[5] L.H. Wiryanto, Small Froude number solutions of flow caused by a line source, *J. of Ind. Math. Soc.* **6**(1):1-6, 2000

[6] Y. Cherruault, Convergence of Adomian's method, *Kybernetes* **18**:31-38, 1989.